

# Ambiguity in risk preferences in robust stochastic optimization

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## Abstract

We consider robust stochastic optimization problems for risk-averse decision makers, where there is ambiguity about both the decision maker's risk preferences and the underlying probability distribution. We propose and analyze a robust optimization problem that accounts for both types of ambiguity. First, we derive a duality theory for this problem class and identify random utility functions as the Lagrange multipliers. Second, we turn to the computational aspects of this problem. We show how to evaluate our robust optimization problem exactly in some special cases, and then we consider some tractable relaxations for the general case. Finally, we apply our model to both the newsvendor and portfolio optimization problems and discuss its implications.

## 1 Introduction

We face two sources of ambiguity in real-world stochastic optimization problems. First, there is ambiguity about the risk preferences of the decision maker. A risk-averse decision maker can express his risk preferences by specifying an increasing and concave utility function. However, it is difficult or impossible to specify an optimal utility function in practice. Instead, we consider the utility function in play to be part of the overall model ambiguity. Second, there is ambiguity about the true underlying probability distribution. This distribution is typically only partially known through historical data or the beliefs of the decision maker. In this paper we focus on risk-aversion in robust stochastic optimization.

Our present paper lies at the intersection between two major streams of research: (i) ambiguity in risk preferences and (ii) robust stochastic optimization. The issue of ambiguity in risk preferences has been thoroughly explored in the stochastic dominance literature. There is a large body of work on stochastic optimization with statistical ambiguity. We mention minimax stochastic programming as considered in [50, 18] as a representative example.

In expected utility maximization, there is considerable ambiguity in the choice of the utility function in practice. There is no obvious choice for the “best” utility function for a single decision maker, and the situation is even more confusing for a group of decision makers trying to agree on a utility function. This concept is closely related to the idea of stochastic dominance, particularly integral stochastic orders (see [41, 48]). The increasing concave stochastic order is defined in terms of all increasing concave functions, and corresponds to the class of risk-averse decision makers. This stochastic order is used to define dominance constraints in optimization in [12, 13], for the univariate case. Extensions to the multivariate case are developed in [15, 16, 31, 32, 29]. As an alternative to stochastic dominance constraints, various minimax expected utility formulations have been devised by [3, 34], and [33]. The formulation in [3] leads to linear programming problems for finite probability spaces when the relaxation of increasing concave stochastic dominance is appropriately chosen.

Robust stochastic optimization was developed for decision makers facing such statistical uncertainty. Conic programming is used to efficiently solve a worst-case value-at-risk portfolio optimization problem in [24]. We also mention robust chance constraints [8] and [7]. Two-stage stochastic linear programming with risk aversion and uncertainty about the underlying probability model is developed in [5]. A new notion of robust stochastic optimization is presented in [4], by allowing the decision maker to vary the protection level over the uncertainty set. This approach is shown to be closely connected to convex risk measures. We refer

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to [11], [44], [10], [37], [19], [9], [43], and [42] for more on robust stochastic optimization with explicit risk preferences. Robust risk measures are developed in [21] which consider the worst-case risk over a family of probability distributions. Models and algorithms for robust multi-objective optimization are developed in [40], [20], and [22]. For more information, a recent survey on robust optimization can be found in [23].

We see that there has been significant work on both ambiguity in risk preferences and robust stochastic optimization, but so far these two problems have been treated separately. We want to combine these two developments into a single robust optimization problem. In addition to handling statistical uncertainty, we would like to express ambiguity about our risk-preferences in stochastic optimization. [17] is the most closely related paper to our present objectives. In [17], robust stochastic dominance constraints are proposed and analyzed. Arguably, the optimization problem in [17] accounts for both ambiguity in the probability model and utility function in play: it requires stochastic dominance to hold over a range of probability distributions, and stochastic dominance itself is a comparison over a range of utility functions. However, the model in [17] is a constrained optimization problem. It is difficult to ensure that an instance of this class of problems will even be feasible, and no solution algorithm is provided.

We make three main contributions with this paper. First, we propose an attractive class of robust optimization problems to simultaneously address these two major types of ambiguity in stochastic optimization. Second, to address practical considerations, we build tractable convex approximations of our notion of regret. Finally, we develop newsvendor and portfolio optimization applications and discuss their managerial implications.

The paper is organized as follows. In Section 2 we develop our methodological framework for the problem of ambiguity in both risk preferences and the probability distribution. We review preliminary material on comparing two random variables under uncertainty. Then, we present our main optimization problem which accounts for model ambiguity in both the underlying probability distribution and the decision maker's risk preferences. In Section 3 we turn to the practical aspects of this problem and develop a computational recipe, including tractable approximations. Then, in Section 4 we apply our methodology to the newsvendor and portfolio optimization problems. The paper concludes in Section 5 with a discussion of managerial insight.

## 2 Methodological framework

This section first introduces our model for ambiguity and then develops the corresponding robust optimization problem. Let  $(\Omega, \mathcal{A})$  be a measure space where  $\Omega$  is a sample space and  $\mathcal{A}$  is a  $\sigma$ -algebra on  $\Omega$ . We let  $\mathbb{L}$  be the linear space of all  $\mathcal{A}$ -measurable mappings  $X : \Omega \rightarrow \mathbb{R}^n$  for  $n \geq 1$ , and we let  $\mathcal{L} \subset \mathbb{L}$  be a particular admissible space of mappings. Let  $\mathbb{Y}$  be the space of signed finite measures on  $(\Omega, \mathcal{A})$ , in the total variation norm:

$$\|\nu\|_{\mathcal{M}(\Omega)} \triangleq \int_{\Omega} d|\nu|.$$

We let  $\mathbb{P} \triangleq \{\nu \in \mathbb{Y} : \|\nu\|_{\mathcal{M}(\Omega)} = 1, \nu \geq 0\} \subset \mathbb{Y}$  be the set of probability distributions on  $\Omega$ , where  $\nu \geq 0$  is understood to mean  $\nu(A) \geq 0$  for all  $A \in \mathcal{A}$ . Corresponding to  $\mathcal{L}$ , we define the dual space  $\mathcal{Y} \subset \mathbb{Y}$  such that  $\int_{\Omega} |X|(\omega) |\nu|(d\omega)$  for all  $X \in \mathcal{L}$ . Each  $\nu \in \mathcal{Y}$  defines a linear functional on  $\mathcal{L}$  given by:

$$\langle \nu, X \rangle = \int_{\Omega} X(\omega) \nu(d\omega), \forall X \in \mathcal{L}.$$

When  $\mu \in \mathbb{P}$ , then  $\langle \nu, X \rangle$  is an expectation.

The statistical ambiguity consists of a set  $\mathcal{Q} \subset \mathbb{P}$  that contains the true underlying probability distribution. We have  $\mathcal{Q}$  because the true underlying distribution is only imperfectly known.

The structural uncertainty set reflects ambiguity about the risk preferences of the decision maker. We can only imperfectly elicit the decision maker's risk preferences. Throughout, we focus on reward maximization so larger values of  $X$  are always preferred. Suppose that we have two mappings  $X, Y \in \mathbb{L}$  that we wish to compare. For simplicity we assume that  $X(\omega), Y(\omega) \in W$  for all  $\omega \in \Omega$  for some compact set  $W$ . For a fixed  $\mu \in \mathbb{P}$ ,  $X$  and  $Y$  become random variables on the probability space  $(\Omega, \mathcal{A}, \mu)$ . Let  $\mathcal{C}(W)$  be the space of continuous functions from  $W$  to  $\mathbb{R}$  in the supremum norm, and let

$$\mathbb{U} \triangleq \{u \in \mathcal{C}(W) : u \text{ is increasing and concave}\} \subset \mathcal{C}(W)$$

be the set of all continuous, increasing, and concave functions on  $W$ . A risk-aware decision maker draws his utility function from  $\mathbb{U}$ . For a fixed probability distribution  $P \in \mathbb{P}$  and a fixed utility function  $u \in \mathbb{U}$ , we write expectation of  $u(X)$  with respect to a  $P \in \mathbb{P}$  as

$$\mathbb{E}_\nu[u(X)] = \langle \nu, u(X) \rangle \equiv \int_{\Omega} u([X](\omega)) \nu(d\omega).$$

We let  $\mathcal{U} \subset \mathbb{U}$  denote the structural uncertainty set that contains a range of utility functions that describe the decision maker's risk preferences. To continue, we become more specific about the uncertainty sets  $\mathcal{U}$  and  $\mathcal{P}$  for the remainder of this paper. We base our choice of  $\mathcal{U}$  on [3], the characteristics of  $u \in \mathcal{U}$  are summarized as follows:

- $u(0) = 0$ . We can translate  $\mathcal{U}$  by a positive constant since  $\mathbb{E}[u(G(z)) - u(Y)] = \mathbb{E}[(u+a)(G(z)) - (u+a)(Y)]$  for any  $a \in \mathbb{R}$ .
- $\mathbb{E}[u(X_0) - u(Y_0)] = 1$  for random variables  $X_0$  and  $Y_0$ . The equality constraint  $\mathbb{E}[u(X_0) - u(Y_0)] = 1$  prevents  $u$  with arbitrarily large magnitude from appearing in  $\mathcal{U}$ , and it also excludes the zero function.
- $\mathbb{E}[u(X_i)] \geq \mathbb{E}[u(Y_i)]$  for pairs of random variables  $(X_i, Y_i)$  indexed by  $i \in I$ . One can imagine presenting the decision maker with a series of pairs of random variables  $\{(X_i, Y_i)\}_{i \in I}$ , and the decision maker selects one from each pair over the other. For clarity, we do not construct  $X_0, Y_0$  and  $X_i, Y_i$  for all  $i \in I$  on  $\Omega$ , rather, we assume they are constructed directly on  $\mathbb{R}$ , and that their distributions are known.

For a set of admissible decisions  $Z \subset \mathbb{R}^m$ , we define the mapping  $G : Z \rightarrow \mathcal{L}$  where  $n \geq 1$ . Notice that we allow  $G(z)$  to be vector-valued, our development in this paper applies to the univariate  $n = 1$  and multivariate  $n \geq 2$  settings simultaneously without any special consideration. We let  $G(z, \omega)$  denote the realization of  $G(z)$  on event  $\omega \in \Omega$ . The following assumptions about problem data hold throughout the paper.

**A1**  $Z \subset \mathbb{R}^m$  is convex and closed.

**A2**  $G$  is continuous in  $\mathcal{L}$ , and  $G(z, \omega)$  is concave in  $z \in Z$  for all  $\omega \in \Omega$  (in the vector-valued sense).

We also make a boundedness assumption on  $G$  for technical convenience.

**A3** There is a compact set  $W \subset \mathbb{R}^n$  such that  $G(z, \omega) \in W$  for all  $(z, \omega) \in Z \times \Omega$ .

The classical robust problem is

$$\max_{z \in Z} \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu[u(G(z))], \quad (1)$$

which maximizes the worst-case expected utility over all models in  $\mathcal{U}$  and  $\mathcal{Q}$ . However, Problem (1) is overly pessimistic.

## 2.1 Shortfall and regret

We now propose a more flexible model than Problem (1). First, we fix a benchmark random variable  $Y \in \mathcal{L}$  which reflects a desirable performance level. Then, we consider

$$\max_{z \in Z} \psi(G(z)) \triangleq \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu[u(G(z)) - u(Y)], \quad (2)$$

which maximizes the worst-case shortfall in expected utility relative to  $Y$ . When  $\mathcal{Q} = \{\nu\}$  is a singleton, then Problem (2) considers ambiguity in  $u$  only. This formulation has appeared in the stochastic dominance literature. When  $\mathcal{U} = \{u\}$  is a singleton then Problem (2) considers ambiguity in  $\nu$  only. This formulation is the classical robust optimization formulation. When  $\mathcal{U}$  and  $\mathcal{Q}$  are both finite and small, then Problem (2) can be solved directly with standard convex optimization techniques.

Cut generation for Problem (2) is captured by

$$\inf_{(u, \nu) \in \mathcal{U} \times \mathcal{P}} \mathbb{E}_\nu [u(G(z)) - u(Y)]. \quad (3)$$

Problem (3) requires special consideration. The following properties are immediate.

**Proposition 2.1.** (i)  $\psi$  is monotonic.

(ii)  $\psi$  is concave.

(iii) Fix  $z \in Z$ . The objective of Problem (3),  $(u, \nu) \rightarrow \mathbb{E}_\nu [u(G(z)) - u(Y)]$ , is bilinear in  $(u, \nu)$ .

*Proof.* (i) Each  $X \rightarrow \mathbb{E}_P [u(X)]$  is monotonic, and thus the infimum of monotonic functions is monotonic.

(ii) Choose  $z_1, z_2 \in Z$  and  $0 \leq \lambda \leq 1$ , then

$$\begin{aligned} \psi(G(\lambda z_1 + (1-\lambda)z_2)) &\leq \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu [u(\lambda G(z_1) + (1-\lambda)G(z_2)) - u(Y)] \\ &\leq \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu [\lambda u(G(z_1)) + (1-\lambda)u(G(z_2)) - u(Y)] \\ &\leq \lambda \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu [u(G(z_1)) - u(Y)] + (1-\lambda) \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{P}} \mathbb{E}_\nu [u(G(z_2)) - u(Y)]. \end{aligned}$$

(iii) Notice that  $(u, \nu) \rightarrow \mathbb{E}_\nu [u(G(z)) - u(Y)]$  is bilinear. For fixed  $\nu \in \mathbb{P}$ ,  $u_1, u_2 \in \mathbb{U}$ , and  $\alpha_1, \alpha_2 \in \mathbb{R}$ , we have

$$\mathbb{E}_\nu [[\alpha_1 u_1 + \alpha_2 u_2](G(z)) - [\alpha_1 u_1 + \alpha_2 u_2](Y)] = \alpha_1 \mathbb{E}_\nu [u_1(G(z)) - u_1(Y)] + \alpha_2 \mathbb{E}_\nu [u_2(G(z)) - u_2(Y)],$$

by linearity of the integral. Similarly, for fixed  $u \in \mathbb{U}$ ,  $\nu_1, \nu_2 \in \mathbb{P}$ , and  $\alpha_1, \alpha_2 \in \mathbb{R}$ , we have

$$\mathbb{E}_{[\alpha_1 \nu_1 + \alpha_2 \nu_2]} [u(G(z)) - u(Y)] = \alpha_1 \mathbb{E}_{\nu_1} [u(G(z)) - u(Y)] + \alpha_2 \mathbb{E}_{\nu_2} [u(G(z)) - u(Y)],$$

by linearity of measures in  $\mathcal{M}(\Omega)$ . □

To continue, we define the new benchmark

$$\vartheta(u, \nu) \triangleq \max_{z \in Z} \mathbb{E}_\nu [u(G(z))], \quad \forall (u, \nu) \in \mathcal{U} \times \mathcal{Q},$$

and introduce the regret minimization problem

$$\max_{z \in Z} \rho(z) \triangleq \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \{\mathbb{E}_\nu [u(G(z))] - \vartheta(u, \nu)\}. \quad (4)$$

Problem (4) does not require a user-defined benchmark  $Y$ , rather it uses the problem data directly. We similarly see that Problem (4) is a convex optimization problem.

## 2.2 Alternative formulations

We briefly comment on some alternative robust formulations that fall within our framework. Suppose  $\mathbb{E}_\nu [u(Y)] \geq \kappa > 0$  for all  $(u, \nu) \in \mathcal{U} \times \mathcal{Q}$  so that the following is well-defined. The worst-case relative shortfall is

$$\max_{z \in Z} \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu [u(G(z))] / \mathbb{E}_\nu [u(Y)] \quad (5)$$

which maximizes the worst-case relative shortfall relative to  $Y$ . We can rewrite Problem (5) as

$$\max_{z \in Z, s \in \mathbb{R}} \{s : \mathbb{E}_P [u(G(z))] \geq \mathbb{E}_P [u(Y)] s, \forall (u, P) \in \mathcal{U} \times \mathcal{Q}\}$$

to reveal that it is a convex optimization problem.

Similarly, we can define

$$\max_{z \in Z} \inf_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_\nu [u(G(z))] / \vartheta(u, \nu), \quad (6)$$

which maximizes the worst-case relative regret.

For a concave function  $f : Z \rightarrow \mathbb{R}$ , we can propose the constrained problem

$$\sup_{z \in Z} \{f(z) : \psi(G(z)) \geq 0\}. \quad (7)$$

Problem (7) is a convex optimization problem under our standing assumptions. This concept is closely related to robust stochastic dominance:

$$\mathbb{E}_\nu[u(G(z))] \geq \mathbb{E}_\nu[u(Y)], \forall (u, \nu) \in \mathcal{U} \times \mathcal{Q}, \quad (8)$$

developed in [17]. Robust stochastic dominance combines the usual notion of stochastic dominance with robustness against ambiguity in the underlying probability distribution. Robust stochastic dominance constraints are developed in [17] for univariate stochastic dominance. When  $\mathcal{P}$  is a singleton then condition (8) is equivalent to the usual notion of stochastic dominance.

### 2.3 Duality

We develop a duality theory for Problem (2) in this subsection that takes advantage of the parametrized representation of uncertainty  $\mathcal{U} \times \mathcal{P}$ . In previous work on stochastic dominance constrained optimization ([12, 13, 29]), utility functions emerged as the Lagrange multipliers of stochastic dominance constraints. We will see that utility functions play a role in duality for our Problem (2) as well, though the presence of the statistical uncertainty set  $\mathcal{P}$  introduces novelty.

We assume that

$$\mathcal{U} = \{u_\xi \in \mathbb{U} : \xi \in \Xi\}$$

is parametrized. We introduce some additional assumptions for the remainder of this paper.

**A4**  $\Xi$  is compact.

**A5**  $u_\xi(w)$  is continuous in  $\xi$  for all  $w \in W$ .

**A6**  $\Omega$  is finite.

For example, in [17], where  $G(z)$  is a univariate random variable,  $\mathcal{U}$  is the collection  $\{\min\{x - \eta, 0\} : \eta \in [a, b]\}$ . This choice is based on the fact that the preceding family is a “generator” of the increasing concave stochastic order (see [41]). From now on, we assume that  $\Omega$  is finite. We will write  $\nu(\{\omega\})$  as a lower case  $p_\omega$  for all  $\omega \in \Omega$  to emphasize that the probability measure is finite-dimensional. We let  $\mathcal{P}$  denote the corresponding set of statistical ambiguity.

In this setting, Problem (2) becomes

$$\max_{z \in Z} \min_{(\xi, p) \in \Xi \times \mathcal{P}} \left\{ \sum_{\omega \in \Omega} p_\omega (u_\xi(G(z, \omega)) - u_\xi(Y(\omega))) \right\}, \quad (9)$$

where we see that the inner minimization problem has finitely many variables. Problem (9) is still a convex optimization problem.

To proceed with a duality theory, we first rewrite Problem (2) as the constrained optimization problem

$$\max_{z \in Z, s \in \mathbb{R}} \left\{ s : \sum_{\omega \in \Omega} p_\omega u_\xi(G(z, \omega)) - \sum_{\omega \in \Omega} p_\omega u_\xi(Y(\omega)) \geq s, \forall (\xi, p) \in \Xi \times \mathcal{P} \right\} \quad (10)$$

by introducing an auxiliary variable  $s \in \mathbb{R}$ . Problem (10) is a convex optimization problem under our assumptions, and it is equivalent to Problem (9) in the sense that both problems have the same optimal value and an optimal solution of one can be recovered from the other.

The constraints in Problem (10) cause Lagrange multipliers to appear. To identify the correct space for the Lagrange multipliers, define the mapping

$$[h(z)](\xi, p) = \sum_{\omega \in \Omega} p_\omega u_\xi(G(z, \omega)) - \sum_{\omega \in \Omega} p_\omega u_\xi(Y(\omega)), \forall (\xi, p) \in \Xi \times \mathcal{P}.$$

We establish continuity of this mapping in the next lemma, let  $\mathcal{C}(\Xi \times \mathcal{P})$  be the space of continuous functions on  $\Xi \times \mathcal{P}$  in the supremum norm, and let  $\mathcal{C}_+(\Xi \times \mathcal{P})$  be the nonnegative cone in  $\mathcal{C}(\Xi \times \mathcal{P})$ .

**Lemma 2.2.**  $h : Z \rightarrow \mathcal{C}(\Xi \times \mathcal{P})$ .

*Proof.* Fix  $z \in Z$ , then we see that  $[h(z)](\xi, p)$  is continuous in  $(\xi, p)$  since it is the sum of the continuous functions  $\sum_{\omega \in \Omega} p_\omega u_\xi(G(z, \omega))$  and  $\sum_{\omega \in \Omega} p_\omega u_\xi(Y(\omega))$ .  $\square$

Based on continuity of  $h$ , the constraint in Problem (10) is equivalent to the conic constraint

$$h(z) \in \mathcal{C}_+(\Xi \times \mathcal{P}).$$

Both  $\Xi$  and  $\mathcal{P}$  are compact Euclidean sets by construction, so we identify  $\mathcal{M}(\Xi \times \mathcal{P})$ , the space of finite signed measures on  $\Xi \times \mathcal{P}$ , as the dual space to  $\mathcal{C}(\Xi \times \mathcal{P})$ . We also define  $\mathcal{M}_+(\Xi \times \mathcal{P})$  to be the space of nonnegative measures on  $\Xi \times \mathcal{P}$ . For  $\Lambda \in \mathcal{M}(\Xi \times \mathcal{P})$ , the Lagrangian for Problem (10) is

$$L(z, s, \Lambda) = s + \int_{\Xi \times \mathcal{P}} ([h(z)](\xi, p) - s) \Lambda(d(\xi, p)).$$

The next theorem computes the dual to Problem (10). The feasible dual variables in the upcoming dual will have a special interpretation.

**Theorem 2.3.** (i) *The dual to Problem (10) is*

$$\min_{\Lambda \in \mathbb{P}} d(\Lambda) \triangleq \max_{z \in Z} \int_{\Xi \times \mathcal{P}} [h(z)](\xi, p) \Lambda(d(\xi, p)). \quad (11)$$

(ii) *Strong duality holds between Problem (10) and Problem (11), i.e. they have the same optimal value.*

*Proof.* (i) Problem (10) is equivalent to

$$\max_{z \in Z, s \in \mathbb{R}} \min_{\Lambda \in \mathcal{M}_+(\Xi \times \mathcal{P})} L(z, s, \Lambda),$$

and interchanging max and min gives the dual problem

$$\min_{\Lambda \in \mathcal{M}_+(\Xi \times \mathcal{P})} \max_{z \in Z, s \in \mathbb{R}} L(z, s, \Lambda).$$

The inner maximization above is explicitly

$$\begin{aligned} & \max_{z \in Z, s \in \mathbb{R}} \left\{ s + \int_{\Xi \times \mathcal{P}} ([h(z)](\xi, p) - s) \Lambda(d(\xi, p)) \right\} \\ &= \max_{z \in Z} \left\{ \int_{\Xi \times \mathcal{P}} [h(z)](\xi, p) \Lambda(d(\xi, p)) \right\} + \max_{s \in \mathbb{R}} \left\{ s \left( 1 - \int_{\Xi \times \mathcal{P}} \Lambda(d(\xi, p)) \right) \right\}, \end{aligned}$$

and the second term is equal to infinity unless  $\int_{\Xi \times \mathcal{P}} \Lambda(d(\xi, p)) = 1$ .

(ii) It is automatic that Problem (10) satisfies the Slater constraint qualification since  $s \in \mathbb{R}$  is unrestricted. In particular, for any  $\hat{z} \in Z$ , we can simply choose  $\hat{s} < \psi(G(\hat{z}))$  to get a Slater point.  $\square$

The preceding theorem reveals that the dual variables in Problem (11) are probability distributions over  $\Xi \times \mathcal{P}$ . In the next theorem we show how to interpret feasible dual variables for Problem (11) in general.

**Theorem 2.4.** *Suppose  $\Lambda \in \mathcal{M}_+(\Xi \times \mathcal{P})$  with  $\int_{\Xi \times \mathcal{P}} \Lambda(d(\xi, p)) = 1$ , then:*

(i) *There exists a probability distribution  $\hat{\Lambda} \in \mathcal{M}_+(\mathcal{P})$  with  $\int_{\mathcal{P}} \hat{\Lambda}(dp) = 1$  and a stochastic kernel  $\phi(\cdot | p)$  from  $\mathcal{P}$  to  $\Xi$  such that*

$$\int_{\Xi \times \mathcal{P}} [h(z)](\xi, p) \Lambda(d(\xi, p)) = \int_{\mathcal{P}} \int_{\Xi} [h(z)](\xi, p) \phi(d\xi | p) \hat{\Lambda}(dp).$$

(ii) *For  $p \in \mathcal{P}$ ,*

$$\int_{\Xi} [h(z)](\xi, p) \phi(d\xi | p) = [h(z)](\xi^*, p) = \mathbb{E}_\nu[\hat{u}_p(G(z)) - \hat{u}_p(Y)],$$

where  $\nu(\{\omega\}) = p_\omega$  for all  $\omega \in \Omega$  and  $\hat{u}_p(w) = \int_{\Xi} u_\xi(w) \phi(d\xi | p)$ .

(iii) It follows that

$$\int_{\Xi \times \mathcal{P}} [h(z)](\xi, p) \Lambda(d(\xi, p)) = \int_{\mathcal{P}} \mathbb{E}_{\mathcal{P}} [\hat{u}_p(G(z)) - \hat{u}_p(Y)] \hat{\Lambda}(dp).$$

*Proof.* (i) For any fixed  $\Lambda$ , we view  $(\Xi \times \mathcal{P}, \mathcal{B}(\Xi \times \mathcal{P}), \Lambda)$  as a probability space, and  $[h(z)](\xi, p) : \Xi \times \mathcal{P} \rightarrow \mathbb{R}$  as a random variable on this probability space. We can then disintegrate the measure  $\Lambda$  into the marginal  $\hat{\Lambda}$  on  $\mathcal{P}$  defined by

$$\hat{\Lambda}(B) = \int I\{u \in B\} \Lambda(d(\xi, p)), \forall B \in \mathcal{B}(\mathcal{P}),$$

and the stochastic kernel  $\phi(\cdot | p)$  which depends on  $p$  (see [45, Appendix F, Theorem 1] for example).

(ii) For fixed  $p \in \mathcal{P}$ , we have

$$\begin{aligned} & \int_{\Xi} [h(z)](\xi, p) \phi(dv | p) \\ &= \int_{\Xi} \sum_{\omega \in \Omega} p_\omega [u_\xi(G(z, \omega)) - u_\xi(Y(\omega))] \phi(d\xi | p) \\ &= \sum_{\omega \in \Omega} p_\omega \int_{\Xi} [u_\xi(G(z, \omega)) - u_\xi(Y(\omega))] \phi(d\xi | p), \end{aligned}$$

by interchanging the order of integration and summation. We recover the concave function  $w \rightarrow \int_{\Xi} u_\xi(w) \phi(d\xi | p)$  citing the fact that  $\phi$  is a transition kernel, and the nonnegative sum of increasing and concave functions is increasing and concave.

(iii) Follows by substituting part (ii) into part (i).  $\square$

Based on the preceding result, we can interpret the expression  $\int_{\Xi \times \mathcal{P}} [h(z)](\xi, p) \Lambda(d(\xi, p))$  as an expectation over expected utilities via  $\int_{\mathcal{P}} \mathbb{E}_p [\hat{u}_p(G(z)) - \hat{u}_p(Y)] \hat{\Lambda}(dp)$ , where the inner utility function depends on  $p \in \mathcal{P}$ .

### 3 Computational considerations

We turn to computational considerations in this section. First, we explain how to solve Problem (2) directly under some special circumstances. Then, more generally we explain how to cast the cut generation for Problem (2) as a bilinear programming problem.

The standard cut generation algorithm is as follows. For a finite set  $\{(\xi_i, p_i)\}_{i \in I} \subset \Xi \times \mathcal{P}$ , we solve

$$\max_{z \in Z} \min_{\{(\xi_i, p_i)\}_{i \in I}} \left\{ \sum_{\omega \in \Omega} p_\omega [u_\xi(G(z, \omega)) - u_\xi(Y(\omega))] \right\}$$

to obtain a candidate solution  $z^* \in Z$ . We choose specific parametrizations for  $\mathcal{U}$  and  $\mathcal{P}$  to discuss the issue of computation. Problem (3) can now be written cleanly (for fixed  $z \in Z$ ) as

$$\min_{(\xi, p) \in \Xi \times \mathcal{P}} \left\{ \sum_{\omega \in \Omega} p_\omega (u_\xi(G(z, \omega)) - u_\xi(Y(\omega))) \right\}. \quad (12)$$

Then, we check if

$$\min_{\{(\xi_i, p_i)\}_{i \in I}} \left\{ \sum_{\omega \in \Omega} p_\omega [u_\xi(G(z, \omega)) - u_\xi(Y(\omega))] \right\} \leq \psi(G(z^*))$$

by exactly evaluating the optimization problem corresponding to  $\psi(G(z^*))$ . If the preceding inequality is violated, then we add the optimal solution  $(\xi^*, p^*) \in \Xi \times \mathcal{P}$  of Problem (12) to  $\{(\xi_i, p_i)\}_{i \in I}$  and repeat the

procedure. Otherwise, we have an optimal solution of Problem (9). It is known that procedures of this type converge for semi-infinite programming problems, see [30, Theorem 7.2] for example. The bottleneck here is the need for repeated calls to a bilinear programming problem, which is generally difficult to solve. We offer later two tractable relaxations of Problem (9) that address this difficulty.

### 3.1 Computation

We use  $\Pr(\cdot)$  to denote the distribution of  $Y$  and  $\{X_i, Y_i\}_{i=0}^r$ . We can now further character the structural uncertainty set  $\mathcal{U}$  as

$$\mathcal{U} = \left\{ u \in \mathbb{U}, \sum_{\theta \in \Theta} \Pr\{X_0 = \theta\} u(\theta) - \sum_{\theta \in \Theta} \Pr\{Y_0 = \theta\} u(\theta) = 1, \right. \\ \left. \sum_{\theta \in \Theta} \Pr\{X_i = \theta\} u(\theta) \leq \sum_{\theta \in \Theta} \Pr\{Y_i = \theta\} u(\theta), \forall i \in I \right\}.$$

We first define

$$\Theta = \text{supp}(Y) \cup \left\{ \bigcup_{i=0}^r \{\text{supp}(X_i) \cup \text{supp}(Y_i)\} \right\} \subset W$$

to be the union of the support of the benchmark  $Y$  and the random variables  $\{X_i, Y_i\}_{i \in I}$  used in the definition of  $\mathcal{U}$ . We make a key assumption about  $\Theta$ .

**A7**  $\Theta$  is finite.

This assumption is not unreasonable since the preceding random variables are all user input. At first glance this representation does not offer any computational advantages. However, for the purposes of solving Problem (3), we can characterize the relevant functions in  $\mathcal{U}$  with finitely many parameters using the method from [6, 3, ?, Section 6.5.5] which specifies convex functions in terms of their subgradients.

We now can specify the parametrization  $\mathcal{U} = \{u_\xi : \xi \in \Xi\}$  and consider for fixed  $p \in \mathcal{P}$  we have

$$\max_{z \in Z} \inf_{(x, p) \in \Xi \times \mathcal{P}} \{ \langle p, \mathbb{G}(z) x \rangle : Ax = b, x \geq 0 \}.$$

Take  $x = (\alpha, \beta, \delta, \gamma, u)$  defined by the following. In this case we have

$$\langle p, \mathbb{G}(z) \xi \rangle = \sum_{\omega \in \Omega} p_\omega [\langle \alpha_\omega, G(z, \omega) \rangle + \beta_\omega - u(Y(\omega))]$$

which is linear in the variables  $\xi$  for fixed  $z \in Z$  with constraints and can be written compactly as a linear system  $A\xi \geq b$ .

$$\langle \alpha_\omega, \theta \rangle + \beta_\omega \geq u(\theta), \quad \forall \omega \in \Omega, \theta \in \Theta, \quad (13)$$

$$\langle \delta_\theta, \theta' \rangle + \gamma_\theta \geq u(\theta), \quad \forall \theta, \theta' \in \Theta, \quad (14)$$

$$\langle \delta_\theta, \theta \rangle + \gamma_\theta = u(\theta), \quad \forall \theta \in \Theta, \quad (15)$$

$$u(0) = 0, \quad (16)$$

$$\sum_{\theta \in \Theta} [\Pr(\{X_0 = \theta\}) - \Pr(\{Y_0 = \theta\})] u(\theta) = 1, \quad (17)$$

$$\sum_{\theta \in \Theta} [\Pr(\{X_i = \theta\}) - \Pr(\{Y_i = \theta\})] u(\theta) \geq 0, \quad \forall i \in I, \quad (18)$$

$$\alpha, \delta \geq 0, \quad (19)$$

To explain, we are specifying the value of a utility function at all  $\theta \in \Theta$ ,  $\{u(\theta)\}_{\theta \in \Theta}$ , and we are specifying two sets of linear functions determined by  $(\alpha, \beta)$  and  $(\delta, \gamma)$  in slope intercept form. Condition (13) simply defines the value of the linear function at  $\theta$ , conditions (14) - (15) are necessary and sufficient conditions

for  $u(\theta)$  to be the values of a concave function (i.e. the subgradient characterization), conditions (17) - (18) restate the refined preference information, and condition (19) requires all linear functions here to be increasing. In particular, we see that constraints (13) - (19) can be written compactly as a linear system  $A\xi \geq b$ .

In general, bilinear programming problems are hard to solve and the solution of Problem (12) requires the solution of several instances of Problem (12). However, the next theorem identifies a special case where it is possible to solve Problem (9) with a single convex optimization problem. In particular, when  $\mathcal{P}$  can be characterized in terms of its extreme points we can solve Problem (9) in one step.

**Theorem 3.1.** *Suppose  $\mathcal{P} = \text{conv}\{\nu_j\}_{j \in J}$ .*

(i) *For fixed  $z \in Z$ ,*

$$\inf_{(\xi, p) \in \Xi \times \mathcal{P}} \mathbb{E}_\nu [u_\xi(G(z)) - u_\xi(Y)] = \inf_{j \in J} \inf_{u \in \mathcal{U}} \mathbb{E}_{\nu_j} [u_\xi(G(z)) - u_\xi(Y)].$$

(ii) *Problem (2) is equivalent to*

$$\max s \tag{20}$$

$$s.t. \langle b, w_j \rangle \geq s, \quad \forall j \in J, \tag{21}$$

$$p_j^T \mathbb{G}(z) \geq w_j^T A, \quad \forall j \in J. \tag{22}$$

*Proof.* (i) First observe

$$\inf_{(\xi, p) \in \Xi \times \mathcal{P}} \mathbb{E}_\nu [u_\xi(G(z)) - u_\xi(Y)] = \inf_{\xi \in \Xi} \inf_{p \in \mathcal{P}} \mathbb{E}_\nu [u_\xi(G(z)) - u_\xi(Y)].$$

The function  $\mathbb{E}_\nu [u(G(z)) - u(Y)]$  is linear in  $\nu$  for fixed  $u \in \mathcal{U}$ , and  $\nu \rightarrow \inf_{u \in \mathcal{U}} \mathbb{E}_\nu [u(G(z)) - u(Y)]$  is concave in  $\nu$  as the infimum of linear functions. By [46, Corollary 32.3.4],

$$\inf_{p \in \mathcal{P}} \inf_{\xi \in \Xi} \mathbb{E}_{\nu_i} [u_\xi(G(z)) - u_\xi(Y)] = \inf_{i \in I} \inf_{\xi \in \Xi} \mathbb{E}_{\nu_i} [u_\xi(G(z)) - u_\xi(Y)],$$

since a concave minimization problem over a polyhedron attains its optimum at one of the extreme points, and  $\{\nu_j\}_{j \in J}$  are the extreme points of  $\mathcal{P}$ .

(ii) By part (i), Problem (9) is equivalent to

$$\max_{z \in Z} \inf_{i \in I} \inf_{\xi \in \Xi} \sum_{\omega \in \Omega} p_\omega [\langle \alpha_\omega, G(z, \omega) \rangle + \beta_\omega - u(Y(\omega))],$$

which can be rewritten as

$$\max_{z \in Z} \left\{ s : s \leq \inf_{\xi \in \Xi} \sum_{\omega \in \Omega} p_j(\omega) [\langle \alpha_\omega, G(z, \omega) \rangle + \beta_\omega - u(Y(\omega))], \forall j \in J \right\}.$$

As shown in [28], for each  $j \in J$  the problem  $\inf_{u \in \mathcal{U}} \mathbb{E}_{P_i} [u(G(z)) - u(Y)]$  is equal to the optimal value of the following linear programming problem. For any  $p \in \mathcal{P}$ , we have

$$\inf_{\xi \in \mathbb{R}^n} \{ \langle p, \mathbb{G}(z) \xi \rangle : A\xi = b, \xi \geq 0 \} = \sup_{w \in \mathbb{R}^q} \{ \langle b, w \rangle : w^T A \leq p^T \mathbb{G}(z) \}$$

by linear programming duality. We use this observation to get the desired result.  $\square$

We see that Problem (20) - (22) is a convex optimization problem in  $z \in Z$ , since the constraints  $p_j^T \mathbb{G}(z) \geq w_j^T A$  are all convex by concavity of  $G$  and nonnegativity of  $p$ . Problem (20) - (22) can be solved directly with standard convex optimization algorithms.

The scale of Problem (20) - (22) grows quickly as the number of extreme points of  $\mathcal{P}$  grows, and we cannot claim that this method is practical for arbitrary  $\mathcal{P}$ . In full generality, we have to treat the inner problem as a bilinear programming problem and use a cut generation algorithm to solve Problem (9).

### 3.2 Convexification

In the last section we saw that Problem (2) can be solved with convex optimization, but that this approach is not scalable when the statistical uncertainty set  $\mathcal{P}$  is complicated. The main difficulty lies with the inner minimization of Problem (2), which is a nonconvex optimization problem. In this section we revise this inner minimization problem to make it convex and obtain a more tractable variant of Problem (2).

The difficulty is the bilinear terms in  $\sum_{\omega \in \Omega} p_{\omega} [\langle \alpha_{\omega}, G(z, \omega) \rangle + \beta_{\omega} - u(Y(\omega))]$ . We showed that evaluating  $\psi(G(z))$  is equivalent to solving a bilinear programming problem where we replace the objective terms  $\sum_{\omega \in \Omega} p_{\omega} \alpha_{\omega}$ ,  $\sum_{\omega \in \Omega} p_{\omega} \beta_{\omega}$ , and  $\sum_{\omega \in \Omega} p_{\omega} u(Y(\omega))$ . We replace the vectors  $p_{\omega} \alpha_{\omega}$  with the variables  $\vartheta_{\omega}$  and the variables  $p_{\omega} \beta_{\omega}$  with the variables  $\zeta_{\omega}$ , for all  $\omega \in \Omega$ . The relaxations will focus on the bilinear terms. There are two main approximation schemes for such constraints in the literature: the reformulation-linearization technique (RLT) and the semidefinite programming (SDP) relaxation. We consider each of these techniques and show how they lead to tractable reformulations of our original Problem (2).

We begin with the following problem

$$\max_{z \in Z} v(G(z)) \triangleq \inf_{(v, p) \in \mathcal{V} \times \mathcal{P}} \{ \langle p, \mathbb{G}_{RLT}(z) v \rangle : C v = d, v \geq 0 \} \quad (23)$$

where we have the bounds. The McCormick inequalities (see [2]). We place bounds  $0 \leq \beta_{\omega} \leq e$  on the slope constraints. Then we obtain Starting with RLT, we place upper and lower bounds on the variables  $\{p_{\omega}\}_{\omega \in \Omega}$ ,  $\{\alpha_{\omega}\}_{\omega \in \Omega}$ , and  $\{\beta_{\omega}\}_{\omega \in \Omega}$  that appear in the nonconvex terms. Recall that we have assumed  $G(z, \omega) \in W$  for all  $z \in Z$  and  $\omega \in \Omega$ . Then necessarily, the intercept terms  $\{\beta_{\omega}\}_{\omega \in \Omega}$  must satisfy  $0 \leq \beta_{\omega} \leq 1$  for all  $\omega \in \Omega$ .

$$\begin{aligned} \vartheta_{\omega} &\geq 0, \forall \omega \in \Omega, \\ 0 &\leq p_{\omega} \leq 1, \forall \omega \in \Omega \\ \vartheta_{\omega} &\geq v_{\omega}^{slp-II} + \beta p_{\omega} e - \beta e, \forall \omega \in \Omega, \\ \zeta_{\omega} &\geq 0, \forall \omega \in \Omega, \\ \zeta_{\omega} &\geq v_{\omega}^{int-II} + p_{\omega} - 1, \forall \omega \in \Omega. \end{aligned}$$

In particular, we can compute the dual of the inner minimization problem to obtain a single maximization problem. The resulting maximization problem turns out to be a convex optimization problem.

**Theorem 3.2.** (i)  $v(G(z)) \leq \psi(G(z))$  for all  $z \in Z$ .

(ii) Problem (23) can be written as

$$\max_{z \in Z, y} \{ \langle d, y \rangle : y^T C \geq \mathbb{H}(z) \},$$

which is a convex optimization problem.

Now we discuss the SDP relaxation

$$\max_{z \in Z} \phi(G(z)) \triangleq \inf_{(w, p) \in \mathcal{W} \times \mathcal{P}} \{ \text{tr}(\mathbb{G}_{SDP}(z) X) : \text{tr}(A_i X) = b_i, \forall i \in I, X \succeq 0 \}. \quad (24)$$

Define

$$\begin{bmatrix} 1 & (p, \alpha, \beta) \\ (p, \alpha, \beta)^T & X \end{bmatrix} \succeq 0.$$

We also use abstract notation in the following theorem to avoid cumbersome details.

**Theorem 3.3.** Problem (24) is equivalent to:

$$\sup_{z \in Z, \lambda} \left\{ \sum_{i \in I} \lambda_i b_i : \mathbb{G}_{SDP}(z) \succeq \sum_{i \in I} \lambda_i A_i \right\}.$$

*Proof.* The Lagrangian of the problem corresponding to  $\phi(G(z))$  is

$$L(X, \lambda) = \text{tr}(\mathbb{G}_{SDP}(z) X) + \sum_{i \in I} \lambda_i (b_i - \text{tr}(A_i X)) = \sum_{i \in I} \lambda_i b_i + \text{tr} \left( \left( \mathbb{G}_{SDP}(z) - \sum_{i \in I} \lambda_i A_i \right) X \right),$$

and the dual functional is

$$d(\lambda) = \begin{cases} \sum_{i \in I} \lambda_i b_i & \mathbb{G}_{SDP}(z) \succeq \sum_{i \in I} \lambda_i A_i, \\ -\infty & \text{otherwise.} \end{cases}$$

The corresponding dual problem is then

$$\phi(G(z)) = \sup_{\lambda} \left\{ \sum_{i \in I} \lambda_i b_i : \mathbb{G}_{SDP}(z) \succeq \sum_{i \in I} \lambda_i A_i \right\}.$$

□

## 4 Practical examples

We develop detailed examples of our method for the classic newsvendor and portfolio optimization problems in this section.

### 4.1 Newsvendor

We first consider the classical newsvendor. We report numerical experiments that demonstrate the importance of *simultaneously* considering ambiguity in risk preferences and ambiguity in the underlying probability distribution. Risk-aware variants of the newsvendor have been extensively studied. A risk-averse expected utility maximizing newsvendor is developed in [36] for uniformly distributed demand, and a closed form optimal solution is computed. Alternatively, [1] considers a coherent risk measure for the classical newsvendor. The conditional value-at-risk minimizing newsvendor is studied in detail in [26] and solved with linear programming. In [51], a variant of the classical newsvendor with risk constraints is proposed and analyzed. The variations of multiplicative risk and random capacity are studied in [47] and [49], respectively.

The newsvendor chooses an order quantity  $q \geq 0$  before seeing the random demand  $D : \Omega \rightarrow \mathbb{R}$  for a single good. Let  $r$  be the unit revenue and  $c$  be the unit order cost, then the random profit is

$$G(q, D) = r \min\{q, D\} - cq.$$

Now let  $\hat{u}$  and  $\hat{\nu}$  be the exact newsvendor's utility function and distribution of demand. Even though the newsvendor cannot observe  $\hat{u}$  and  $\hat{\nu}$ , he has uncertainty sets  $\mathcal{U}$  and  $\mathcal{Q}$  such that  $\hat{u} \in \mathcal{U}$ ,  $\hat{\nu} \in \mathcal{Q}$ . For our experiments, we consider the regret minimization problem

$$\max_{q \geq 0} \min_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_P [u(G(q, D))] - \vartheta(u, \nu), \quad (25)$$

which we compare with the pure robust problem

$$\max_{q \geq 0} \min_{(u, \nu) \in \mathcal{U} \times \mathcal{Q}} \mathbb{E}_{\nu} [u(G(q, D))], \quad (26)$$

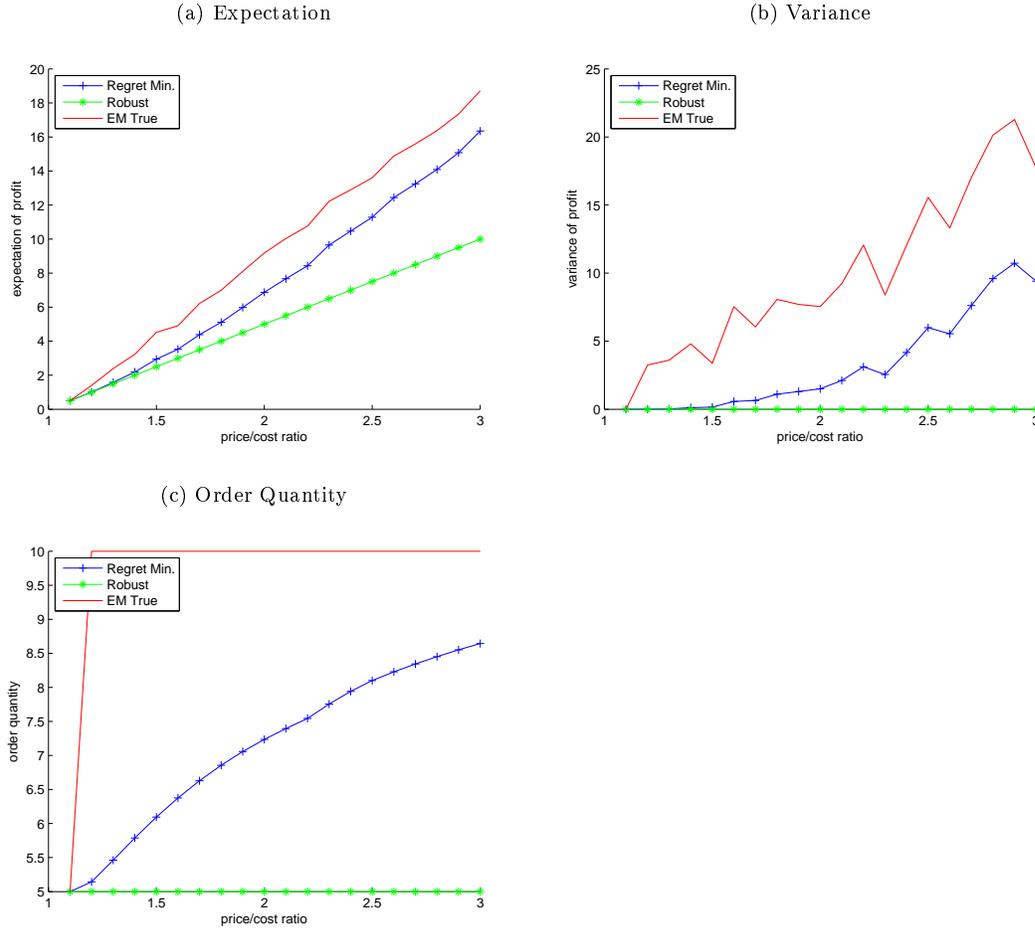
and the classical risk-neutral model

$$\max_{q \geq 0} \mathbb{E}_{\nu} [G(q, D)]. \quad (27)$$

We will compare the expectation  $\mathbb{E}_P [G(q, D)]$  and the variance  $\text{var}(G(q, D)) \triangleq \mathbb{E}_P [(G(q, D) - \mathbb{E}_P [G(q, D)])^2]$  across their different optimal solutions  $q$  when  $\nu$  is taken to be the true model.

For simplicity in our first experiment, we set  $\Omega = \{\omega_1, \omega_2\}$  and  $\mathcal{Q} = \{P_1, P_2\}$ . Detailed parameter settings are as follows:

Figure 1: Bernoulli Distribution with Finite Set



- There are two demand scenarios,  $d_L = D(\omega_1) = 5$  “low demand” and  $d_H = D(\omega_2) = 10$  “high demand”;
- $\mathcal{U} = \{u_1, u_2\}$  and  $\mathcal{Q} = \{P_1, P_2\}$ . We choose  $u_1(x) = 3x^{1/3}$ ,  $u_2(x) = -2e^{-x/10+1}$  (parameters for  $u_2$  are chosen in order to make it comparable to  $u_1$ ),  $P_1(\omega_1) = 0.1$ , and  $P_2(\omega_1) = 0.7$ ;
- $\hat{P} = P_1$  is the true model.

The results of this first experiment are reported in Figure 1. Figure 1a shows how the expected profit changes as the price/cost ratio increases. The variance and the optimal order quantity for each method are also shown as functions of the price/cost ratio in 1b and 1c respectively.

We now try a more general experiment with the following parameters:

- $\mathcal{U} = \{u_1, u_2\}$  where  $u_1(x) = 3x^{1/3}$  and  $u_2(x) = -2e^{-x/10+1}$ ;
- $\Omega = \{\omega_1, \omega_2, \dots, \omega_{10}\}$ ,  $\mathcal{Q} = \{P_1, P_2, \dots, P_5\}$ , and  $\hat{P} = P_1$ ;
- $D(\omega_i) = 10 * i$ ,  $i = 1, 2, \dots, 10$ .

Detailed distributional information is given in table 1, and the results appear in Figure 2.

From Figures 1 and 2, we can see that our regret minimization problem (25) finds a solution that balances expected profit against variance. The classical expectation maximization problem (27) achieves the best performance in terms of maximizing expectation by being overly optimistic in its order quantity, as shown in Figures 1c and 2c, but it also gives the highest variance. In actuality, the performance of the expectation

Table 1: Discrete Distribution Information

	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$
$D(\omega_1)$	0.1812	0.0202	0.1627	0.0013	0.1082
$D(\omega_2)$	0.1549	0.1232	0.0884	0.1087	0.1273
$D(\omega_3)$	0.2085	0.1486	0.1748	0.0336	0.1081
$D(\omega_4)$	0.0251	0.1187	0.1272	0.0500	0.1645
$D(\omega_5)$	0.0911	0.1956	0.0567	0.0207	0.0464
$D(\omega_6)$	0.0189	0.2070	0.0030	0.0389	0.0026
$D(\omega_7)$	0.0494	0.0025	0.1186	0.1753	0.0519
$D(\omega_8)$	0.0442	0.0198	0.0680	0.0026	0.1275
$D(\omega_9)$	0.1093	0.0671	0.1375	0.2845	0.0555
$D(\omega_{10})$	0.1174	0.0973	0.0630	0.2845	0.2079

maximization problem (27) will be worse because we have assumed it has access to the true distribution  $\hat{P}$  to run these experiments. On the other hand, the robust optimization problem (26) makes conservative decisions by ordering the least amount, achieving the smallest variance but also the lowest expected profit.

As the price/cost ratio increases, the difference between Problem (27) and Problem (26) becomes more pronounced. Since Problem (26) optimizes against the worst-case scenario, the order quantity and the variance do not change much, while the expected profit increases. For Problem (27), the order quantity is very sensitive to the price/cost ratio. Both the expectation and the variance increase dramatically as this ratio increases. Our regret minimizing model (25) behaves similarly to (27), but with a smaller decrease in expected profit and a significant improvement in terms of the variance.

We also note that our regret minimizing model provides a smooth transition for the order quantity as the price/cost ratio increases. Due to our assumption of finite  $\Omega$ , the optimal order quantity for Problem (27) is piecewise constant. Intuitively, as the price/cost ratio increases, the order quantity will jump to a higher level since the marginal profit is increasing. On the other hand, these results imply that the order quantity for Problem (26) will first increase as the price/cost ratio increases and then will become relatively stable regardless of the price/cost ratio. In contrast, the order quantity for our model Problem (25) provides a smooth increase as the price/cost ratio increases.

These experimental results imply that our model Problem (25) is appropriate for decision makers who are between the risk averse decision makers of (26) and the risk neutral decision makers of (27). Our model is useful especially when the difference between Problem (27) and Problem (26) is large (as shown in Figure 2c)) since it provides a reasonable way to achieve relatively better profit and relatively lower risk.

For a final experiment, we use the infinite uncertainty sets:

$$\mathcal{U} = \{u \in \mathbb{U} : \mathbb{E}[u(X_0) - u(Y_0)] = 1\},$$

and

$$\mathcal{Q} = \left\{ p \in \mathbb{R}^{|\Omega|} : \sum_{\omega \in \Omega} p_\omega = 1, p \geq 0 \right\},$$

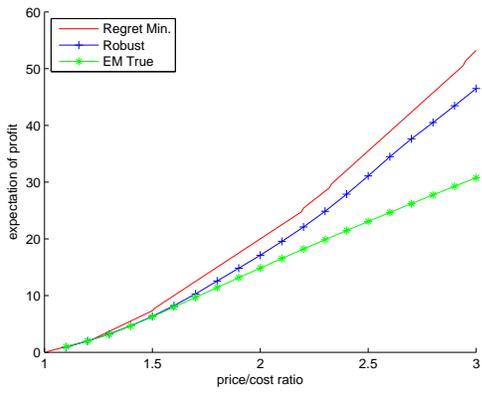
and we take the following parameter settings:

- choose two demand scenarios,  $d_L = D(\omega_1) = 5$  “low” and  $d_H = D(\omega_2) = 10$  “high”;
- let  $G_{\min}(c, r) = r * D(\omega_1) - c * D(\omega_2)$ , which is a lower bound for  $G(q, D)$  given unit revenue  $r$  and unit cost  $c$ . Then, we take  $X_0 = G_{\min}(c, r) - 1$  and  $Y_0 = G_{\min}(c, r) - 2$ , which are constants given  $r$  and  $c$ ;
- take the “true” demand distribution to be  $\hat{P}(\omega_1) = 0.3$  and  $\hat{P}(\omega_2) = 0.7$ .

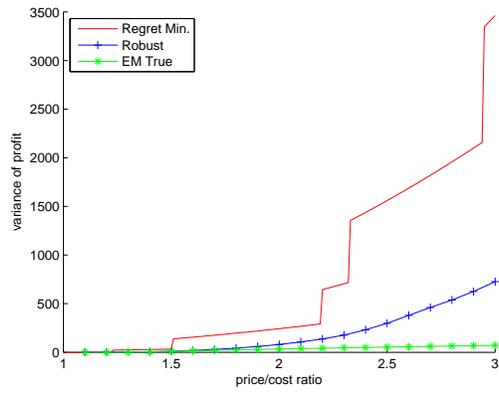
We evaluate the inner minimization in Problems (25) and (26) for any call by using a general nonconvex quadratic programming solver. We solve Problems (25) and (26) themselves using the bisection method.

Figure 2: Discrete Distributions with Finite Set

(a) Expectation



(b) Variance



(c) Order Quantity

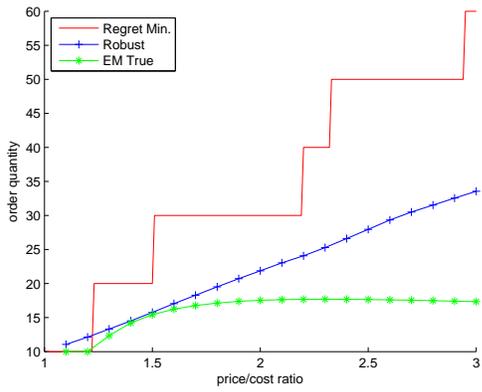


Figure 3: Bernoulli Distribution with Infinite Set

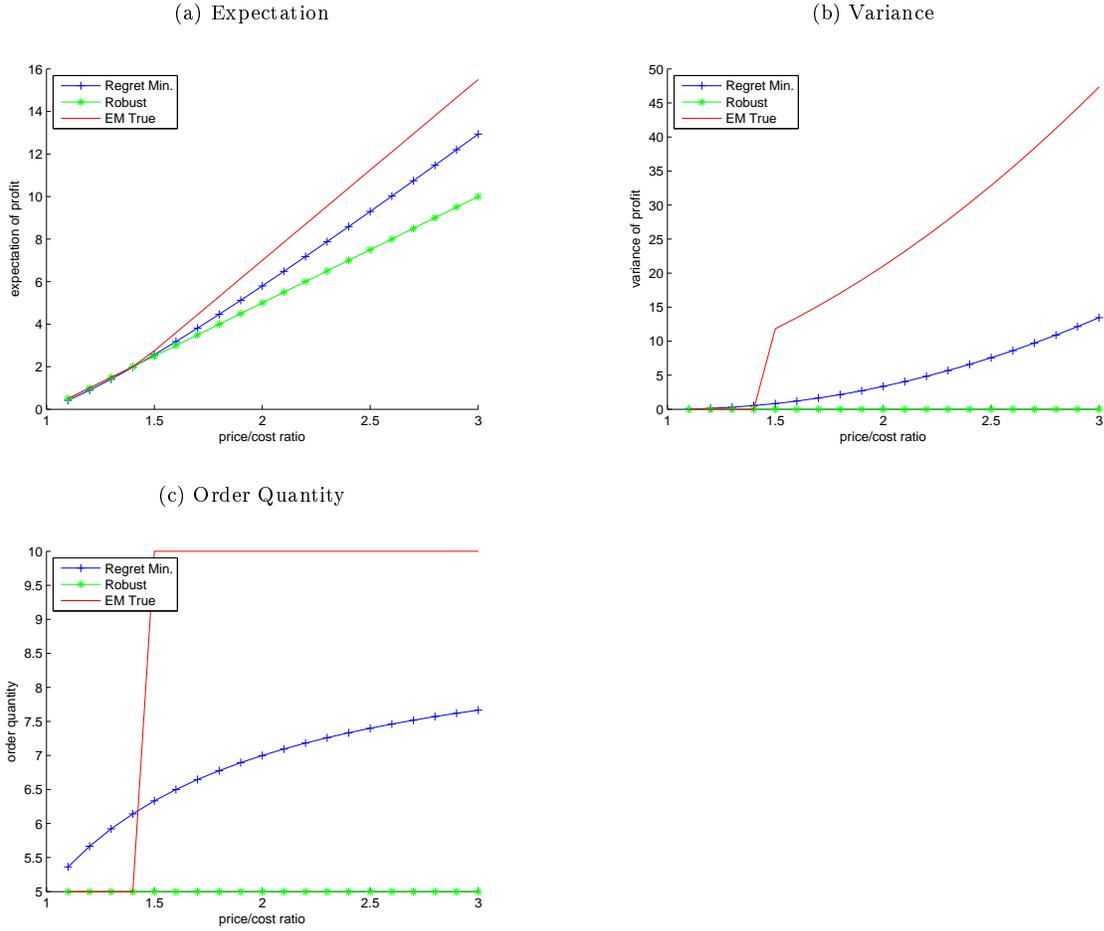


Figure 3 records the results of this experiment. Compared to the results shown in Figure 1, our regret minimization model provides a smoother function for the optimal order quantity as the price-cost ratio increases. We hypothesize that this smoothness is due to the choice of the infinite uncertainty sets. In summary, we see that our method gives a good balance between expected profit and variance with respect to the true distribution.

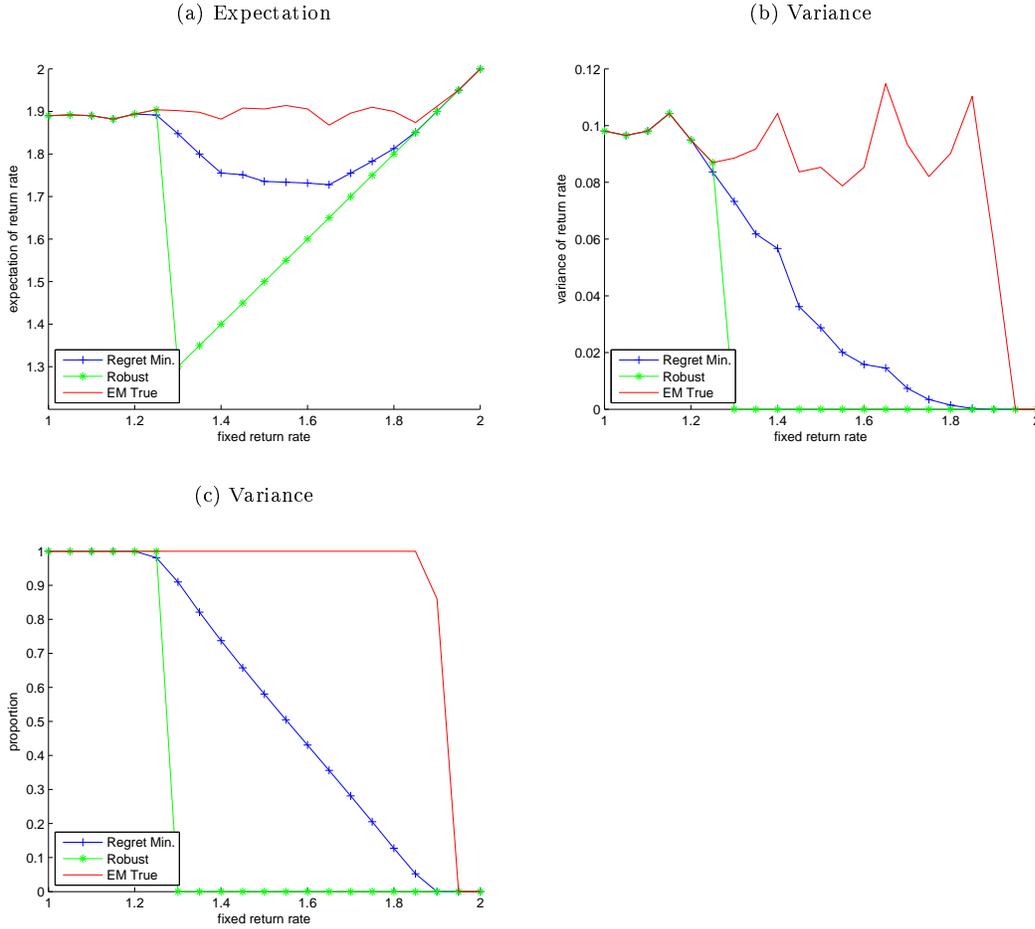
## 4.2 Portfolio optimization

We consider the portfolio optimization problem in this subsection. The issue of risk is naturally quite important in this problem and has been carefully studied. In [14], the stochastic dominance constrained portfolio optimization is studied which emphasizes ambiguity in risk preferences. A robust portfolio optimization problem is formulated in [35] which defines regret in terms of conditional value-at-risk. The papers [38, 39] develop an expected utility regret model for portfolio optimization under statistical ambiguity. Several variations on the classical robust portfolio optimization problem are developed in [52] (insurance guarantees), [27] (cost of robustness), and [25] (asymmetric returns).

We study a simple portfolio optimization problem with two assets: one risky asset with random return rate  $R$  and one riskless asset with constant return rate  $\alpha \geq 1$ . Let  $z \in Z \triangleq \{z \in \mathbb{R} : 0 \leq z \leq 1\}$  be the proportion of total wealth invested in the risk asset, then for  $z \in Z$  the random return is  $Rz + \alpha(1 - z)$ . We set the parameters for these experiments as follows:

- Random return rate  $R$  can take two values 1 and 2;

Figure 4: Discrete Distributions with Finite Set



- $\mathcal{Q} = \{P_1, P_2\}$ ,  $P_1(R = 1) = 0.1$ , and  $P_2(R = 1) = 0.7$ ;
- $\mathcal{U} = \{u_1, u_2\}$ ,  $u_1(x) = 3x^{1/3}$ ,  $u_2(x) = -2e^{-x/10+1}$  (parameters for  $u_2$  are chosen in order to make it comparable to  $u_1$ );
- $\hat{P} = P_1$  is the true model.

We vary the fixed return rate  $\alpha$  between 1 and 2. Figure 4 reports the expectation, variance, and the optimal investment decision as a function of  $\alpha$  for the three different models. Just like for the newsvendor problem, these experimental results show that our regret minimizing model provides a good balance between the expectation maximization approach and the robust optimization approach. It is appropriate for decision makers who fall between pure risk-aversion and risk-neutrality.

## 5 Conclusion

In this paper, we argue that decision makers face both structural uncertainty (ambiguity about risk preferences) and statistical uncertainty (ambiguity about the underlying probability distribution) in practical problems. In particular, it is just as hard to elicit risk preferences as it is to identify the exact probability distribution in play. In response, we incorporate ambiguity in risk preferences into robust stochastic optimization. Most robust stochastic optimization problems emphasize statistical ambiguity only, so our model offers a novel perspective that fills this gap. We give a practical computational recipe for our problem

and demonstrate the worth of our model through experiments for the newsvendor and portfolio optimization problems. Our experiments show that our regret minimizing model gives a good balance between the conservativeness of a pure robust optimization problem and the optimism of risk-neutrality.

For future research, we are interested in an online extension of the methods in this paper. Our present models are all offline optimization problems that do not incorporate new data in real time. An online model would have the same emphasis on ambiguity in risk preferences but would use data acquisition to inform the decision making process (exploration versus exploitation). As a motivating example, exploratory pricing in the newsvendor problem can be modeled as a multi-armed bandit. Each pricing decision leads to an observation from a demand distribution that depends on that pricing decision. Pricing decisions are made to optimize performance while simultaneously learning about the different demand distributions. The usual multi-armed bandit approach aims to find the pricing decision with minimum expected loss. Our goal is to determine ordering and pricing policies with favorable risk profiles.

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