Computationally efficient train timetable generation of metro networks with uncertain transfer walking time to reduce passenger waiting time: A generalized Benders decomposition-based method

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Abstract

With more and more interchange stations in a large-scale metro network, passengers tend to transfer between different metro lines from origination to destination, sometimes even more than once. Passenger waiting time is one of the critical standards for measuring the quality of urban public transport services. To support high service quality, this paper proposes a mixed integer nonlinear programming (MINLP) model for the train timetable generation problem of a metro network that minimizes the transfer waiting times and access passenger waiting times. In the mathematical formulation of the model, the transfer walking times at the interchange stations between two connected lines are treated as uncertain parameters. The robust train timetable generation model is formulated to optimize timetables by adjusting arrival and departure times of each train in the metro network to reduce access and transfer passenger waiting times. A robust counterpart is further derived that transforms the formulated robust model into a deterministic one. Moreover, a generalized Benders decomposition technique based approach is developed to decompose the robust counterpart into a subproblem and a master problem. The subproblem is a convex quadratic programming problem that can be solved efficiently. Finally, two sets of numerical examples, consisting of a small case and a large-scale case based on a real-world portion of the Beijing metro network, are performed to demonstrate the validity and practicability of the proposed model and solution approach.

Keywords: Urban metro network, Timetabling generation, Uncertain transfer time, Robust optimization, Benders decomposition

1. Introduction

With increased urbanization and motorization, problems such as traffic congestion, traffic accidents, and pollution have significantly increased. To alleviate urban traffic congestion and promote environmental protection, prioritizing the development of urban public transportation has become an important measure widely adopted by different countries worldwide. As a critical component of public transportation, urban rail transit is favored by major cities for its large capacity, fast speed, and energy savings. With the increasing expansion of residents' trips and the diversity of travel demands, transfers have significantly influenced urban public transportation. In an urban rail transit network, passengers sometimes need to complete at

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least one transfer trip from origin to destination during their travel (Wu et al., 2015a; Li et al., 2018). The transfer time directly affects the passenger travel times and the passenger satisfaction with the service level of the transportation system. Thus, improving transfer efficiency and reducing transfer waiting times are of great importance to metro networks. However, it is difficult to change the infrastructure, such as route and station construction, in practice to improve the transfer efficiency. Correspondingly, it is more desirable and practical to synchronize each train's arrival and departure times at interchange stations based on the existing infrastructure to reduce the total waiting times for transfer and access passengers.

Timetable synchronization and optimization is an essential topic in public transport network planning. By reducing operational costs and allowing more flexible planning, transfers in public transport are used to create a more efficient network (Ceder et al., 2013), especially in an urban rail transit network system. Its aim is such that two trains on two different lines can arrive at the transfer station synchronously, and passengers can catch the connecting train by walking through the transfer platform. With a synchronized and coordinated schedule, passengers can experience seamless transfer connections and good service of the metro system. Transfer coordination of different metro lines in the metro network has a significant influence on the service quality (Wong et al., 2008; Wu et al., 2015a; Kang et al., 2016). On account of the high train service frequency during peak hours, the maximum transfer synchronization time for saturated lines is close to the relatively short headway. While passengers who transfer among different lines may spend excess waiting time for available trains during off-peak hours with low train service frequency. Furthermore, the metro operators usually adjust train timetables with different scales of headways based on the various travel demands and different periods of the day. Inappropriate coordination among train services on different lines can bring about unreasonable passenger waiting times during the transitional period (from peak to off-peak hours or vice versa) (Guo et al., 2017). Hence, it is important to enhance transfer synchronization and timetable coordination among different metro lines. Specifically, a new timetable needs to be generated in urban transit railway operations due to different passenger flow characteristics of workdays and holidays, or large-scale activities. It however tends to take relatively long solving times to determine new timetables, which is a computational burden for the operating company. To overcome this challenge, it is essential to develop a computationally efficient optimization algorithm for the timetable generation problem of a large-scale metro network.

In this paper, a mixed integer nonlinear programming (MINLP) optimization model is presented to obtain a synchronized and coordinated schedule by adjusting the arrival and departure times of the feeder and connecting trains within reasonable ranges. Moreover, the objective is to adjust the train connections on two connected lines to minimize the waiting times for passengers outside the stations and those transferring at interchange stations. Due to the difficulty of obtaining accurate transfer walking times of each passenger, the transfer walking times here are considered uncertain parameters, and a robust optimization model is proposed. Furthermore, the complexity of large-scale network problems makes it computationally difficult to solve realistic-sized problems. Therefore, we develop a generalized Benders decomposition technique to divide the network problem into a subproblem (SP) and a master problem (MP) to solve the optimization model efficiently.

The rest of this paper is structured as follows. A brief literature review and the main contributions of this paper are presented in Section 2. Next, we describe the timetable coordination and optimization problem in detail and propose a robust optimization model by treating the transfer walking times as uncertain parameters. In Section 4, we design a generalized Benders decomposition based approach to solve the MINLP model. Based on actual operation data of the Beijing metro network, an illustrative small numerical example and a large-scale example are presented in Section 5. Finally, conclusions and discussions for the future are given in Section 6.

2. Literature Review

Timetables of public transport play a vital role in a company's operation efficiency and traveling experience for passengers. Thus, many researchers have focused on timetable synchronization and optimization from various point of views, including level of service, energy consumption, etc (Corman et al., 2012; Cacchiani et al., 2014; Corman et al., 2014; Tian and Niu, 2017; Hu and Hu, 2020; Shen et al., 2020; Li et al., 2021; Wang et al., 2022).

Unreasonable transfers can reduce the attractiveness and competitiveness of public transportation (Muller and Furth, 2009). The timetable optimization problem with transfer connections has attracted many researchers' attention. Tian and Niu (2017) focused on synchronizing the fixed and flexible trains at a rail transfer station to obtain an optimal timetable and adopted an exponential utility function to estimate the train connection quality with the aim of maximizing the seamless train connections and minimizing passenger transfer waiting time. Cao et al. (2019) applied a genetic algorithm with a local search strategy to solve the timetable scheduling synchronization and coordination problem for larger-sized railway networks to maximize the synchronized connections considering smooth transfers at interchange stations.

There is a wealth of literature addressing the timetabling problem aiming to minimize the waiting time for transfer passengers. As a significant topic on bus network planning, timetable synchronization is investigated by many researchers (Chu et al., 2019; Abdolmaleki et al., 2020), especially for considerable bus network planning during off-peak periods. Ibarra-Rojas and Rios-Solis (2012) formulated the network-based timetabling problem to maximize the number of synchronizations. Wu et al. (2016) proposed a multi-objective re-synchronizing of the bus timetable model to balance the number of passengers benefited by smooth transfers and the maximal deviation from the departure times of the existing timetable. Furthermore, Xiong et al. (2016) investigated the problem of optimizing synchronized timetables for community shuttles linked with metro service. Takamatsu and Taguchi (2020) focused on the timetable design problem in areas with low-frequency public transportation services to design a timetable that ensures smooth transfers among buses and trains. Concerning the train timetabling problem, Wong et al. (2008) developed a MIP model for non-periodic timetables, which minimizes the total waiting time for all transfer passengers. They introduced binary variables to represent the waiting time for the "next available" train at the transfer stations and solved it with an optimization-based heuristic algorithm. Wu et al. (2015a) presented a timetable scheduling coordination model to decrease the weighted sum of the probability and propagation of delay as well as the transfer waiting time, considering the headway of the different lines as the decision variables. Wu and Tang (2012) presented a MINLP model and a genetic algorithm solution method for the public transport schedule synchronization problem to minimize transfer waiting times. Shi et al. (2016) focused on the timetable optimization problem for a loop line and adjusted the headway and dwell time with the aim of minimizing the average waiting time for access and transfer passengers. They solved the model by applying a genetic algorithm.

Furthermore, some papers have focused on the first and last train timetabling problem considering transfer coordination. Guo et al. (2016) put forward a timetable synchronization optimization model to minimize the connection time according to the importance of transfer stations by adjusting the departure times of the first train on each line to avoid the just-missed situation. They demonstrated their approach using data from the Beijing railway network. Similarly, Kang et al. (2016) proposed an optimization model to minimize the amount of missed trains and train arrival time differences. Li and Shi (2013) presented a method of compiling the last train timetables of all the lines at the network level considering transfer coordination and put forward a pick-and-filter algorithm for choosing connection principles based on passenger transfer demands. Chen et al. (2019) proposed a MIP model and a modified genetic algorithm to decide the arrival and departure times of the last trains and maximize the linear weighted sum of accessible origin-destination (OD) pairs for last trains to enhance the metro network accessibility. The results of the case based on the Shenzhen metro network showed that a larger quantity of successful transfer train connections do not necessarily result in better accessibility.

Urban rail transit organization is a systematic program comprised of several stages, including network design, route planning, train timetabling, rolling stock, and staffing (Kang et al., 2016). Therefore, some researchers studied the integration and collaborative optimization of some of these problems in recent years. Cheng et al. (2018) investigated the timetabling and network design problem simultaneously for an urban public transport system and solved the MIP model using a parallel branch-and-price-and-cut (BPC) algorithm. Blanco et al. (2020) presented a MILP model to find solutions to line planning and timetabling problems considering important factors, including time-dependent demands and interchange stations. Fonseca et al. (2018) integrated the timetabling and vehicle scheduling problem by modifying the scheduled timetables to minimize transfer and operational costs. Liu and Ceder (2018) proposed a bi-objective and bi-level integer programming model and considered the operators to solve timetable synchronization and optimization integrated with vehicle scheduling. In addition, Yang et al. (2016a) developed a coordination and optimization model to integrate the train scheduling and stops planning problem on a single-track high-speed railway corridor and formulated a multi-objective MILP model by means of linear weighted methods.

The real world has to deal with uncertainty, such as train running times, dwell times, headways, transfer walking times, the number of passengers boarding and alighting, and weather conditions. Wu et al. (2015b) presented a stochastic integer programming model with random travel times for the timetabling problem in order to minimize the waiting time costs of three kinds of passengers consisting of passengers boarding, through, and transferring at stations, and designed a genetic algorithm with a local search strategy. Liu and Dessouky (2019) considered the uncertainty on departure times of the freight trains with the passenger train timetabling problem and proposed a two-stage stochastic optimization model to minimize the operation costs and solved the model using a branch and bound framework with hybrid heuristics. Yang et al. (2016b) presented a two-phase stochastic programming model and designed a genetic algorithm based on simulation for the integrated speed profile and timetable optimization problem to minimize the total tractive energy consumption allowing for uncertain train mass. Zhang et al. (2020) investigated the design of timetables to determine a series of time intervals adopted to the dynamic passenger flow by introducing two fuzzy variables (i.e., passenger satisfaction and vehicle capacity usage). Cheng et al. (2018) investigated a three-echelon logistics network and solved three two-stage robust models using a column-and-constraint-generation algorithm.

Due to the explosion in size and complexity of modern datasets, researchers applied decomposition methods for a number of transportation problems, such as network-level traffic signal control (Mohebifard and Hajbabaie, 2019), and location problem (Arslan and Karaşan, 2016; Tapia-Ubeda et al., 2018; Mahéo et al., 2019) to reduce the computation complexity. Liu and Ceder (2016) studied the synchronization timetabling problem of public transport routes by integrating several factors, including headways and vehicle trip offset times. They proposed a two-objective function and a decomposition method to minimize the observed difference on passenger load and the expected passenger waiting time. Kang et al. (2019) presented a last train and bus bridging coordination MILP model and an effective decomposition approach for handling large-scale problems, which decomposed the original MILP into two smaller MILP models. Liu et al. (2020) investigated a collaborative optimization problem considering train scheduling, train connections, and passenger flow control strategy as a MINLP model and designed a Lagrangian relaxation-based solution method.

Based on the above literature, Table 1 summarizes some publications closely related to the timetable coordination problem and this paper, including the type of model, the objective, and solution approaches. To enhance the service level and attractiveness of the metro system, this paper presents a MINLP model aimed at minimizing the transfer waiting time and waiting time for access passengers entering the station from outside the station. Specifically, the main contributions of this paper are as follows:

(1) Compared to the model proposed by (Wong et al., 2008; Wu et al., 2015a) whose objective is only to minimize passenger transfer waiting time, the timetable coordination optimization model constructed in this paper minimizes the transfer waiting time and access passenger waiting time to improve the operational efficiency. By altering the weights of these two terms in the objective function, the metro operating company can balance these two kinds of waiting times. In addition, our formulation introduces binary variables to model constraints subject to transfers, arrival times and departure times, and headways explicitly.

(2) Compared to most of the above papers, this paper considers the uncertainty of the transfer walking times between two connected lines at the interchange stations. The formulated robust model is transformed into a deterministic model to obtain robust solutions, where the conservatism level of the solutions can be adjusted by changing the robustness parameters.

(3) It is challenging to derive reasonable solutions within an acceptable computation time due to the complexity of large-scale mixed integer programming models. To achieve computationally efficient train timetable generation of a large-scale metro network, this paper applies a generalized Benders decomposition technique to split the metro network problems into a subproblem with a convex quadratic programming problem and a master problem to significantly improve the solution efficiency.

Table 1: Comparison of recent publications on the timetable coordination problem and this paper

Publication	Type of Model	Objective	Solution Approach
Wong et al. (2008)	0-1 MILP	Minimize the total transfer waiting time	Optimization-based heuristic method
Wu and Tang (2012)	0-1 MINLP	Minimize the total transfer waiting time	Genetic algorithm
Wu et al. (2015b)	Stochastic integer programming model	Minimize the total waiting time cost	Genetic algorithm with local search
Wu et al. (2015a)	0-1 MILP	Minimize the maximal passenger waiting time	Genetic algorithm
Shi et al. (2016)	0-1 MINLP	Minimize the average waiting time of the total boarding passengers	Genetic algorithm
Guo et al. (2016)	0-1 MINLP	Maximize the transfer synchronization events	Hybrid algorithm based on the Particle Swarm Optimization and Simulated Annealing (PSO-SA)
Cao et al. (2019)	0-1 MILP	Maximize synchronized meetings	Genetic algorithm with a local search strategy
Liu et al. (2020)	0-1 MILP	Train connection, passenger control strategy and number of stranded passengers	Lagrangian relaxation based algorithm
This paper	0-1 MILP	Minimize waiting time of transfers and passengers at the non-interchange stations	Benders decomposition technique

3. The timetable generation problem

In this section, the timetable generation problem for metro networks is defined by considering safety, operation, and service quality concerning passenger waiting time. The passenger waiting time includes transfer waiting time at the interchange stations and waiting time for access passengers outside the metro network. The constraints on timetables and passenger transfer waiting times are constructed by introducing binary variables.

3.1. Model assumptions

The assumptions of the timetable generation problem are as follows.

Assumption 1. The transfer choices of passengers are assumed to be known and fixed. For the frequent and uniform metro services, passengers are more concerned with the waiting time, and they value their waiting time significantly more than the on-board running time (Goodman and Murata, 2001; Hollander and Liu, 2008). Based on this, we assume that passengers prefer not to transfer so that they would choose a path with as few interchanges as possible (Wong et al., 2008), in the sense of less transfer waiting time. The passengers are assumed to choose their paths by two criteria: the number of interchanges and the number of stops during the trip. Passengers choose a path with as few interchanges as possible. When the alternative paths take the same number of interchanges, passengers would choose the one with fewer stops. This assumption allows us to compute the number of transfer passengers at each interchange station based on origin-destination counts more easily. However, it cannot guarantee the minimum travel time. Moreover, other more accurate methods can be used to calculate the number of transfer passengers, such as the minimum travel time. It does not affect the structure of our mathematical model but only changes the coefficients in the objective function.

Assumption 2. To easily calculate the transfer waiting time and access passenger waiting time, we assume that the train's capacity can accommodate all passengers who want to get on the train. The oversaturated condition tends to occur during the peak period (e.g., 08:00-09:00 in the Beijing metro). However, the train capacity is sufficient in other periods, such as the transitional and off-peak periods. This assumption applies to most periods when the train capacity is sufficient. To incorporate the oversaturated condition during peak hours and relax this assumption in the proposed model, one may choose the proper weighting coefficients in the objective function for the number of transfer passengers and average access passenger arrival rates to calculate the passenger waiting time more realistically. For example, during the peak hours of the oversaturated station, the weighting coefficient of the number of transfer passengers includes not only the transfer passengers for the current train but also the transfer passengers for the ahead train who failed to board it due to the capacity limitation.

Assumption 3. Each train is assumed to depart from its initial station and finish its service without turning around. All trains run in a first-in-first-out manner without overtaking and crossing operations.

3.2. Mathematical formulations

The variables and parameters are listed in Table 2 for formulating the timetable optimization model.

Table 2: Variables and parameters used in the paper

M	the set of operation lines.
m, m'	index of different lines.
s, s'	index of stations on different lines.
S_m	the set of stations of line m.
i, i'	index of trains on the different lines.
N_m	the set of trains to be scheduled on line m.
T	the set of transfer stations between two different lines, $(m, m', s) \in T$.
System Parameters	
$e_{m,s}^{m'}$	nominal walking time for passengers transfering from line m' to m at transfer station s.
$\hat{e}_{m,s}^{m'}$	maximum deviation of walking time for passengers transfering from line m' to m at transfer station s.
$\widetilde{e}_{m,s}^{m'}$	uncertain walking time for passengers transfering from line m' to m at transfer station s.
$c_{m,s}^{m',i'}$	the number of transfer passengers from train i' on line m' to line m at transfer station s.
$b_{i,s}^{m}$	average passenger arrival rate that passengers enter station s and wait for train i on line m.
$H_{\min}^{m,s}, H_{\max}^{m,s}$	the minimum and maximum headway at station s of line m.
$D_{\min}^{m,s}, D_{\max}^{m,s}$	the minimum and maximum dwell time at station s of line m.
$R_{\min}^{m,s}, R_{\max}^{m,s}$	the minimum and maximum running time between station s and $s+1$ on line m.
$T_{\min}^{m}, T_{\max}^{m}$	the minimum and maximum total trip time of line m.
G	the planning horizon.
Decision Variables	
$A_{i,s}^{m}$	the arrival time of train i at station s on line m.
$L_{i,s}^{m}$	the departure time of train i at station s on line m.
$R_{i,s}^{m}$	the running time of train i from station s to $s+1$ on line m.
$D_{i,s}^{m}$	the dwell time of train i at stations s on line m.
$\alpha_{m,i,s}^{m',i'}$	0-1 binary variable, if train i' on line m' arrives at transfer station s early enough so that passengers can transfer to train i on line m, $\alpha_{m,i,s}^{m',i'} = 1$; otherwise, $\alpha_{m,i,s}^{m',i'} = 0$.
$\beta_{m,i,s}^{m',i'}$	0-1 binary variable, if train i' on line m' can successfully connect with train i on line m at transfer station s, $\beta_{m,i,s}^{m',i'} = 1$; otherwise, $\beta_{m,i,s}^{m',i'} = 0$.

3.2.1. Constraints for timetable

To ensure the continuity of train operation, the arrival time $A_{i,s}^{m}$ of train i on line m at station s is the sum of the departure time $L_{i,s-1}^{m}$ of train i at station $s-1$ on line m and the running time $R_{i,s-1}^{m}$ of train i between station $s-1$ and s on line m, as shown in constraint (1). Constraint (2) demonstrates the departure time of train i at station s on line m, and $D_{i,s}^{m}$ represents the dwell time of train i at station s on line m. For each route $m \in M$, station $s \in S_m$ and train $i \in N_m$,

$$A_{i,s}^{m} = L_{i,s-1}^{m} + R_{i,s-1}^{m}, \qquad \forall m \in M, s-1, s \in S_m, i \in N_m. \tag{1}$$

$$L_{i,s}^{m} = A_{i,s}^{m} + D_{i,s}^{m}, \qquad \forall m \in M, s \in S_m, i \in N_m. \tag{2}$$

Different from a single metro line where the running time between two adjacent stations is usually the same, the speed of trains running in the same segment could be different for better transfer coordination in the case of a large-scale urban transit network problem. For instance, feeder train i' on line m' may run faster than the previous train to catch the connecting train i on line m at transfer station s in the metro network. By making the running time for each train between stations as variables, the proposed model can be more generalized to cope with different situations. Specifically, the running time for trains in the same segment can be set equivalent if necessary.

Constraint (3) ensures the headway at station s on line m is between the upper and lower bounds. $L^m_{i,s} - L^m_{i-1,s}$ is defined as the headway, which means the difference between the departure time of adjacent trains i and $i-1$ at the same station s on line m. For the sake of meeting operation safety and service requirements, we define $H^{m,s}_{\min}$ and $H^{m,s}_{\max}$ as the minimum and maximum headway at station s on line m respectively. For each route $m \in M$, station $s \in S_m$ and train $i \in N_m$,

$$H^{m,s}_{\min} \le L^m_{i,s} - L^m_{i-1,s} \le H^{m,s}_{\max}, \qquad \forall m \in M, s \in S_m, i-1, i \in N_m. \tag{3}$$

Constraint (4) sets the upper and lower bounds of the total travel time on line m (i.e., $T^m_{\min}$ and $T^m_{\max}$) and it ensures the time for train i from its arrival at the last station $A^m_{i,last}$ to its departure from the first station $L^m_{i,1}$ on line m is within a reasonable total trip time. For each route $m \in M$, $s \in S_m$ and $i \in N_m$,

$$T^m_{\min} \le A^m_{i,last} - L^m_{i,1} \le T^m_{\max}, \qquad \forall m \in M, i \in N_m. \tag{4}$$

In addition, the dwell times and running times have to satisfy safety and operation needs, constraints (5) and (6) enforce the given upper and lower limits. The minimum and maximum dwell time and running time are introduced as $D^{m,s}_{\min}, D^{m,s}_{\max}$ and $R^{m,s}_{\min}, R^{m,s}_{\max}$ respectively. For each route $m \in M$, $s \in S_m$ and $i \in N_m$,

$$D^{m,s}_{\min} \le D^m_{i,s} \le D^{m,s}_{\max}, \qquad \forall m \in M, s \in S_m, i \in N_m. \tag{5}$$

$$R^{m,s}_{\min} \le R^m_{i,s} \le R^{m,s}_{\max}, \qquad \forall m \in M, s \in S_m, i \in N_m. \tag{6}$$

For each route $m \in M$, constraint (7) ensures all trains complete their trips within the planned horizon G, and $A^m_{last,last}$ means the arrival time of the last train at the terminal station on line m.

$$0 \le A^m_{last,last} \le G, \qquad \forall m \in M. \tag{7}$$

3.2.2. Constraints for transfer waiting time

The transfer waiting time here is defined as the minimum possible transfer time based on the given assumptions. The time $e^{m'}_{m,s}$ that passengers alighting from the feeder train on line m' to the boarding area of connecting trains on line m at the transfer station s is related to the walking transfer distance and train dwelling time (Du et al., 2009). Especially in crowded conditions, the passenger walking times are different because they are affected by pushing or squeezing and their physical characteristics. Therefore, the transfer walking times are taken as uncertain parameters here.

In the case of line m and line m' intersecting at station s, if the transfer behaviors of passengers are feasible from line m' to m, it is necessary that when passengers get off train i' on line m' and walk to the platform on line m, train i has not left the platform of transfer station s. To represent the train connections at the interchange stations, binary variable $\alpha^{m',i'}_{m,i,s}$ is defined as the following constraint (8).

$$M(\alpha^{m',i'}_{m,i,s} - 1) \le L^m_{i,s} - A^{m'}_{i',s} - \widetilde{e}^{m'}_{m,s} \le M\alpha^{m',i'}_{m,i,s} \qquad \forall (m, m', s) \in T, i \in N_m, i' \in N_{m'}. \tag{8}$$

where M is a large positive number and $\widetilde{e}^{m'}_{m,s}$ refers to the uncertain transfer walking times. $A^{m'}_{i',s} + \widetilde{e}^{m'}_{m,s}$ is the earliest time that passengers arriving at interchange station s from feeder train i' on line m' can transfer successfully and $L^m_{i,s}$ is the departure time of the connecting train i on line m. Note that if $\alpha^{m',i'}_{m,i,s} = 1$, then $L^m_{i,s} - A^{m'}_{i',s} - \widetilde{e}^{m'}_{m,s} \geq 0$; it means feeder train i' on line m' arrives early enough while the connecting train i on line m departs late enough at transfer station s and passengers can catch the connecting train. If $\alpha^{m',i'}_{m,i,s} = 0$, then $L^m_{i,s} - A^{m'}_{i',s} - \widetilde{e}^{m'}_{m,s} < 0$, and the case is opposite to the previous.

Based on the above definition of $\alpha^{m',i'}_{m,i,s}$, $\beta^{m',i'}_{m,i,s}$ is formulated as constraint (9).

$$\beta^{m',i'}_{m,i,s} = \alpha^{m',i'}_{m,i,s} - \alpha^{m',i'}_{m,i-1,s} \qquad \forall (m,m',s) \in T, i-1, i \in N_m, i' \in N_{m'}. \tag{9}$$

where $\beta^{m',i'}_{m,i,s}$ represents the actual connections between two lines according to Assumption 2 that there are no passenger capacity limitations and all the passengers get on the first available train. If train i on line m, rather than train $i-1$, is the first available train to connect with train i' on line m', we obtain $\beta^{m',i'}_{m,i,s} = 1$ and $\beta^{m',i'}_{m,i+1,s} = 0$ based on constraint (9). According to the meaning of $\alpha^{m',i'}_{m,i,s}$, it holds that if $\alpha^{m',i'}_{m,i,s} = 0$, one has $\alpha^{m',i'}_{m,i-1,s} = 0$, i.e., $\beta^{m',i'}_{m,i,s} = 0$. Similarly, if $\alpha^{m',i'}_{m,i,s} = 1$, one has $\alpha^{m',i'}_{m,i-1,s} = 1$ or $\alpha^{m',i'}_{m,i-1,s} = 0$, i.e., $\beta^{m',i'}_{m,i,s}$ takes a value of 0 or 1. Thus, $\beta^{m',i'}_{m,i,s}$ only takes a value of 0 or 1.

On account of constraints (8) and (9), if train i on line m is the first connection with train i' on line m', it is equivalent to the following equations. For each transfer pair $(m,m',s) \in T$, $\beta^{m',i'}_{m,i,s}$ and $\alpha^{m',i'}_{m,i,s}$ satisfy the following equation,

$$\beta^{m',i'}_{m,i,s} = \begin{cases} 1, & \sum\limits_{(m,m',s)\in T} \sum\limits_{i' \in N_{m'}} \sum\limits_{i \in N_m} \alpha^{m',i'}_{m,i,s} > 0, \\ 0, & \sum\limits_{(m,m',s)\in T} \sum\limits_{i' \in N_{m'}} \sum\limits_{i \in N_m} \alpha^{m',i'}_{m,i,s} = 0. \end{cases} \tag{10}$$

3.2.3. Objective functions

The objective of our model is to minimize the passenger waiting time, including the transfer waiting time and the waiting time for access passengers outside the metro network. Thus, the objective function is indicated as follows:

$$Q = \sum_{(m,m',s)\in T} \sum_{i' \in N_{m'}} \sum_{i \in N_m} \rho_1 c^{m',i'}_{m,s} \left(L^m_{i,s} - A^{m'}_{i',s} - \widetilde{e}^{m'}_{m,s} \right) \beta^{m',i'}_{m,i,s} + \sum_{m \in M} \sum_{i \in N_m} \sum_{s \in S_m} \rho_2 \frac{b^m_{i,s}}{2} (L^m_{i,s} - L^m_{i-1,s})^2, \tag{11}$$

where $c^{m',i'}_{m,s}$ is the transfer passenger demands in trains i' on line m' aiming for line m at transfer station s, and $b^m_{i,s}$ is the average passenger arrival rate for passengers entering station s and waiting for train i on line m. The first term represents the waiting time for all the transfer passengers. If feeder train i' on line m' can connect with train i on line m successfully, the transfer waiting time can be calculated as $L^m_{i,s} - A^{m'}_{i',s} - \widetilde{e}^{m'}_{m,s} \geq 0$, while the transfer waiting time from train i' on line m' to the other trains equals 0. The second term of the objective represents the waiting time for access passengers outside the metro network. The number of passengers arriving at the stations is assumed to satisfy a uniform distribution (Niu et al., 2015) during the time interval $L^m_{i,s} - L^m_{i-1,s}$, where $b^m_{i,s}$ is the average passenger arrival rate, which can be estimated from historical statistical data. In particular, for variant headway, we use the average passenger arrival rate to represent the number of passengers arriving at stations during the time interval to calculate the passenger waiting time easily. The term $b^m_{i,s}(L^m_{i,s} - L^m_{i-1,s})$ represents the average number of access passengers. Thus, the total waiting time of access passengers during the time interval $L^m_{i,s} - L^m_{i-1,s}$ equals the average number of access passengers $b^m_{i,s}(L^m_{i,s} - L^m_{i-1,s})$ multiplied by the average waiting time for each passenger $\frac{1}{2}(L^m_{i,s} - L^m_{i-1,s})$, i.e., $\frac{b^m_{i,s}}{2}(L^m_{i,s} - L^m_{i-1,s})^2$. In addition, there exists globally optimal solution to the convex quadratic program, which can be solved quickly using existing optimization tools. Minimizing the transfer waiting time could increase the connections of trains and lengthen the dwell time and headway at the expense of the travel experience of the passengers who do not need to transfer. The pre-specified weights ρ_1 and ρ_2 can balance the transfer waiting time and the access passenger waiting time for the metro network.

In light of frequent and uniform metro services with a large number of commuters, passengers are more concerned with the waiting time (including the transfer waiting time and the waiting time for access passengers outside the metro network) instead of in-vehicle time. The waiting time has become an important evaluation criterion for the passenger service quality of metro networks. Therefore, for the high-frequent metro networks with abundant commuters, it is reasonable to assume that passengers are not especially conscious of the timetable, and passengers' transfer choice behaviors are not sensitive to minor changes in the timetable. Based on this, the numbers of transfer passengers among different lines $c^{m',i'}_{m,s}$ are predetermined constants based on the original timetable (Wong et al., 2008; Fonseca et al., 2018; Abdolmaleki et al., 2020), which are regarded as weighting coefficients in the objective function. Because the formulated train timetable scheduling model is a planning model, the model and method presented in this paper are effective and easy to implement with this assumption. To incorporate the case that passengers cannot transfer successfully for the first time, which usually occurs during peak hours, an appropriate weighting coefficient in the objective function for the number of transfer passengers should be chosen to calculate the passenger waiting time more realistically. For example, for the oversaturated interchange stations during peak hours, the weighting coefficient of the number of transfer passengers includes not only the transfer passengers for the current train but also the transfer passenger who failed to board the previous train due to the capacity limitation, which can be estimated by historical statistics. Then, along with the Assumption 2 that all the passengers who want to transfer can get on the first connecting train, and under constraints (8) and (9), we can obtain that $\sum_{i \in N_m} \beta^{m',i'}_{m,i,s} = 1$ for any given $c^{m',i'}_{m,s}$ only if there exists a successfully connected train i on line m at transfer station s, $i \in N_m$ (i.e., $L^m_{i,s} - A^{m'}_{i',s} - e^{m'}_{m,s} > 0$). Correspondingly, the transfer waiting time for all the transfer passengers $c^{m',i'}_{m,s}$ to the successfully connected train i are counted in the objective function, which is $c^{m',i'}_{m,s}\left(L^m_{i,s} - A^{m'}_{i',s} - e^{m'}_{m,s}\right)$. Therefore, the minimization of objective $\sum_{(m,m',s)\in T} \sum_{i' \in N_{m'}} \sum_{i \in N_m} \beta^{m',i'}_{m,i,s} c^{m',i'}_{m,s} \left(L^m_{i,s} - A^{m'}_{i',s} - e^{m'}_{m,s}\right)$ implies to choose a better connected train i on line m with less transfer waiting time for the given number of transfer passengers $c^{m',i'}_{m,s}$. For example, for a larger number of transfer passengers $c^{m',i'}_{m,s}$, we should reduce the term $L^m_{i,s} - A^{m'}_{i',s} - e^{m'}_{m,s}$ under $\beta^{m',i'}_{m,i,s} = 1$, i.e., adopt a shorter time interval between the departure time of successfully connected train i on line m and the arrival time of feeding train i' on line m' at transfer station s for the given parameter $e^{m'}_{m,s}$, so as to achieve the minimization of total transfer waiting time, which is in accordance with the practice. The first term of the objective function is to minimize the waiting time of all the successfully transferring passengers. In addition, the objective to maximize the number of successful transfer connections between different lines has been taken into account for the last train timetabling optimization problem (Kang et al., 2015a,b), which is out of the scope of this paper.

3.3. Timetable coordination optimization model

With objective function (11) and the constraints of arrival and departure times (1)-(2), transfer waiting time constraints (8) - (9) and bound constraints (3)-(7), the complete timetable coordination optimization model is constructed as follows.

$$\text{Min} \quad Q = \sum_{(m,m',s)\in T} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \rho_1 c_{m,s}^{m',i'} \left(L_{i,s}^m - A_{i',s}^{m'} - \widetilde{e}_{m,s}^{m'} \right) \beta_{m,i,s}^{m',i'} + \sum_{m\in M} \sum_{i\in N_m} \sum_{s\in S_m} \rho_2 \frac{b_{i,s}^m}{2} (L_{i,s}^m - L_{i-1,s}^m)^2$$

$$\text{s.t.} \begin{cases} A_{i,s}^m = L_{i,s-1}^m + R_{i,s-1}^m, & \forall m \in M, s-1, s \in S_m, i \in N_m, \\ L_{i,s}^m = A_{i,s}^m + D_{i,s}^m, & \forall m \in M, s \in S_m, i \in N_m, \\ H_{\min}^{m,s} \le L_{i,s}^m - L_{i-1,s}^m \le H_{\max}^{m,s}, & \forall m \in M, s \in S_m, i-1, i \in N_m, \\ D_{\min}^{m,s} \le D_{i,s}^m \le D_{\max}^{m,s}, & \forall m \in M, s \in S_m, i \in N_m, \\ R_{\min}^{m,s} \le R_{i,s}^m \le R_{\max}^{m,s}, & \forall m \in M, s \in S_m, i \in N_m, \\ T_{\min}^{m,s} \le L_{i,last}^m - A_{i,1}^m \le T_{\max}^{m,s}, & \forall m \in M, i \in N_m, \\ 0 \le A_{last,last}^m \le G, & \forall m \in M, \\ M(\alpha_{m,i,s}^{m',i'} - 1) \le L_{i,s}^m - A_{i',s}^{m'} - \widetilde{e}_{m,s}^{m'} \le M\alpha_{m,i,s}^{m',i'}, & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ \beta_{m,i,s}^{m',i'} = \alpha_{m,i,s}^{m',i'} - \alpha_{m,i-1,s}^{m',i'}, & \forall (m,m',s) \in T, i-1, i \in N_m, i' \in N_{m'}. \end{cases} \tag{12}$$

The above robust optimization model is an uncertain mixed integer nonlinear programming problem whose objective function is nonlinear, where $\widetilde{e}_{m,s}^{m'}$ is an uncertain parameter. Since the constraints with uncertain parameters have to be satisfied for any feasible values, this optimization problem includes an infinite number of constraints, which leads to an intractable problem in general. To handle this issue, the robust counterpart of the model is further derived in the next section based on a robust optimization method.

3.4. Robust counterpart of the optimization model

Since the transfer walking time $\widetilde{e}_{m,s}^{m'}$ is uncertain in the proposed timetable coordination optimization model, the robust counterpart of the model (12) is developed in this section. We describe the robust timetable optimization model as a maximum and minimum optimization model and convert it into a deterministic model using the duality theory proposed by Bertsimas and Sim (2004).

The uncertainty of the transfer passenger walking time between two lines at the transfer stations can be defined as $\widetilde{e}_{m,s}^{m'} \in \left[e_{m,s}^{m'} - \hat{e}_{m,s}^{m'}, e_{m,s}^{m'} + \hat{e}_{m,s}^{m'} \right]$, where $e_{m,s}^{m'}$ is the nominal value of the transfer walking time and $\hat{e}_{m,s}^{m'}$ represents the maximum fluctuation. Let $J_{m,s}^{m'}$ be the set of the coefficients $\widetilde{e}_{m,s}^{m'}$ which are subject to parameter uncertainty. The parameter $\Gamma_{m,s}^{m'}$ taking a value in the interval $\left[0, |J_{m,s}^{m'}|\right]$ is introduced to regulate the degree of disturbance of the transfer walking time of each interchange station and control the level of conservatism of the robust model. $\Gamma_{m,s}^{m'}$ for all the interchange stations in the metro network satisfy the following constraint,

$$\sum_{(m,m',s)\in T} \Gamma_{m,s}^{m'} = \Gamma. \tag{13}$$

Generally speaking, it means the transfer walking times of up to Γ interchange nodes (there are two interchange nodes between two connecting lines) on the metro network can be at their worst simultaneously. The decision makers predetermine the value of Γ, reflecting their risk preferences. If the decision maker is more conservative, Γ is set as a relatively large value. The part of the model (12) with uncertain parameters is shown as follows.

$$\text{Min} \quad Q_r = \sum_{(m,m',s)\in T} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \rho_1 c_{m,s}^{m',i'} \left(-\widetilde{e}_{m,s}^{m'} \right) \beta_{m,i,s}^{m',i'}$$

$$\text{s.t.} \begin{cases} M(\alpha_{m,i,s}^{m',i'} - 1) - L_{i,s}^m + A_{i',s}^{m'} + \widetilde{e}_{m,s}^{m'} \le 0, & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ -M\alpha_{m,i,s}^{m',i'} + L_{i,s}^m - A_{i',s}^{m'} - \widetilde{e}_{m,s}^{m'} \le 0, & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}. \end{cases} \tag{14}$$

The robust optimization method aims to optimize the objective function under the worst case of uncertain parameters (which is the Min-Max problem here). It seeks a solution that is not necessarily optimal but must be feasible for arbitrary values of uncertain parameters. It is challenging to solve the above model, and we transform (14) into a robust counterpart that can be solved directly, presented as follows.

$$\begin{aligned}
\text{Min} \quad Q_r = \text{Max} \quad -Q_r = & \sum_{(m,m',s)\in T} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \rho_1 c_{m,s}^{m',i'} e_{m,s}^{m'} \beta_{m,i,s}^{m',i'} \\
+\min_{\boldsymbol{S}} \sum_{i'\in N_{m'}} \sum_{i\in N_m} & \left\{ \sum_{s\in K} \left(\rho_1 c_{m,s}^{m',i'} \hat{e}_{m,s}^{m'} \beta_{m,i,s}^{m',i'} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \rho_1 c_{m,t_s}^{m',i'} \hat{e}_{m,t_s}^{m'} \beta_{m,i,t_s}^{m',i'} \right) \right\} \\
= & \sum_{(m,m',s)\in T} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \rho_1 c_{m,s}^{m',i'} e_{m,s}^{m'} \beta_{m,i,s}^{m',i'} \\
-\max_{\boldsymbol{S}} \sum_{i'\in N_{m'}} \sum_{i\in N_m} & \left\{ \sum_{s\in K} \left(\rho_1 c_{m,s}^{m',i'} \hat{e}_{m,s}^{m'} \beta_{m,i,s}^{m',i'} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \rho_1 c_{m,t_s}^{m',i'} \hat{e}_{m,t_s}^{m'} \beta_{m,i,t_s}^{m',i'} \right) \right\}
\end{aligned}$$

$$\text{s.t.} \begin{cases} M(\alpha_{m,i,s}^{m',i'} - 1) - L_{i,s}^m + A_{i',s}^{m'} + e_{m,s}^{m'} + \max_{\boldsymbol{S}} \left\{ \hat{e}_{m,s}^{m'} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \hat{e}_{m,t_s}^{m'} \right\} \leq 0, \\ \qquad \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ -M\alpha_{m,i,s}^{m',i'} + L_{i,s}^m - A_{i',s}^{m'} - e_{m,s}^{m'} + \max_{\boldsymbol{S}} \left\{ \hat{e}_{m,s}^{m'} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \hat{e}_{m,t_s}^{m'} \right\} \leq 0, \\ \qquad \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}. \end{cases} \tag{15}$$

where the subset $\boldsymbol{S} = \left\{ K_{m,s}^{m'} \cup \{t_{m,s}^{m'}\} | K_{m,s}^{m'} \subseteq J_{m,s}^{m'}, |K_{m,s}^{m'}| = \lfloor \Gamma_{m,s}^{m'} \rfloor, t_{m,s}^{m'} \in J_{m,s}^{m'} \setminus K_{m,s}^{m'} \right\}$. $J_{m,s}^{m'}$ is the set of the coefficients $\widetilde{e}_{m,s}^{m'}$ which are subject to parameter uncertainty and $K_{m,s}^{m'}$ is the set of the uncertain parameters $\widetilde{e}_{m,s}^{m'}$ which are changed. $\lfloor \cdot \rfloor$ means the largest integer less than this number. Then, up to $\lfloor \Gamma_{m,s}^{m'} \rfloor$ of these coefficients are allowed to change, and one coefficient $e_{m,t_s}^{m'}$ changes by $(\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \hat{e}_{m,t_s}^{m'}$.

The following propositions are presented to further reformulate the model (15) into a linear programming model.

Proposition 1. Given the optimal vector $\boldsymbol{\beta^*}$, the protection function

$$\pi_1(\boldsymbol{\beta^*}, \Gamma_{m,s}^{m'}) = \max_{\boldsymbol{S}} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \left\{ \sum_{s\in K} \left(\rho_1 c_{m,s}^{m',i'} \hat{e}_{m,s}^{m'} \beta_{m,i,s}^{m',i'*} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \rho_1 c_{m,t_s}^{m',i'} \hat{e}_{m,t_s}^{m'} \beta_{m,i,s}^{m',i'*} \right) \right\}, \tag{16}$$

in (15) is equivalent to the following linear programming problem by introducing auxiliary variables $p_{m,s}^{m'}$ and $X_{m,s}^{m'}$, where $\beta_{m,i,s}^{m',i'*}$ is the optimal solution to the timetable optimization model (12).

$$\min \sum_{(m,m',s)\in T} p_{m,s}^{m'} + \Gamma_{m,s}^{m'} X_{m,s}^{m'}$$

$$\text{s.t.} \begin{cases} X_{m,s}^{m'} + p_{m,s}^{m'} \geq \sum_{i'\in N_{m'}} \sum_{i\in N_m} (\rho_1 c_{m,s}^{m',i'} \beta_{m,i,s}^{m',i'}) \hat{e}_{m,s}^{m'}, & \forall (m,m',s) \in T, \\ p_{m,s}^{m'} \geq 0, & \forall (m,m',s) \in T, \\ X_{m,s}^{m'} \geq 0, & \forall (m,m',s) \in T. \end{cases} \tag{17}$$

Proof. In order to transform the nonlinear term (16) into a linear program for ease of solving, the following linear programming problem is constructed by introducing variables $z_{m,s}^{m'}$ according to the proposition by Bertsimas and Sim (2004).

$$\pi_1(\boldsymbol{\beta^*}, \Gamma_{m,s}^{m'}) = \max \sum_{(m,m',s)\in T} \left(\sum_{i'\in N_{m'}} \sum_{i\in N_m} \rho_1 c_{m,s}^{m',i'} \beta_{m,i,s}^{m',i'*} \right) \hat{e}_{m,s}^{m'} z_{m,s}^{m'}$$

$$\text{s.t.} \begin{cases} z_{m,s}^{m'} \leq \Gamma_{m,s}^{m'}, & \forall (m,m',s) \in T, \\ 0 \leq z_{m,s}^{m'} \leq 1, & \forall (m,m',s) \in T. \end{cases} \tag{18}$$

The optimal solution for variables $z_{m,s}^{m'}$ to the above problem (18) contains $\lfloor \Gamma_{m,s}^{m'} \rfloor$ variables equal to 1 and only one variable equal to $(\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor)$ (Bertsimas and Sim, 2004). The solution is equivalent to the selection of the subset $\boldsymbol{S} = \left\{ K_{m,s}^{m'} \cup \{t_{m,s}^{m'}\} | K_{m,s}^{m'} \subseteq J_{m,s}^{m'}, |K_{m,s}^{m'}| = \lfloor \Gamma_{m,s}^{m'} \rfloor, t_{m,s}^{m'} \in J_{m,s}^{m'} \setminus K_{m,s}^{m'} \right\}$, corresponding to the objective function $\sum\limits_{s \in K} \sum\limits_{i' \in N_{m'}} \sum\limits_{i \in N_m} \rho_1 c_{m,s}^{m',i'} \beta_{m,i,s}^{m',i'} \hat{e}_{m,s}^{m'} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \rho_1 c_{m,t_s}^{m',i'} \beta_{m,i,t_s}^{m',i'} \hat{e}_{m,t_s}^{m'}$. Hence, function (16) equals the objective of the linear programming problem (18) with variables $z_{m,s}^{m'}$.

If $\Gamma = 0$, which means $\Gamma_{m,s}^{m'} = 0, \forall (m, m', s) \in T$, then $\widetilde{e}_{m,s}^{m'} = e_{m,s}^{m'}$ and $\pi_1(\boldsymbol{\beta^*}, \Gamma_{m,s}^{m'}) = 0$, the model (14) is equivalent to the deterministic model when the transfer walking time is at the nominal value $e_{m,s}^{m'}$. Let P be the sum of the transfer nodes in the entire metro network. If $\Gamma = P$, which means $\Gamma_{m,s}^{m'} = |J_{m,s}^{m'}|, \forall (m, m', s) \in T$, then $\widetilde{e}_{m,s}^{m'} \in \left[e_{m,s}^{m'} - \hat{e}_{m,s}^{m'}, e_{m,s}^{m'} + \hat{e}_{m,s}^{m'} \right]$ and all the transfer walking times are uncertain parameters. Note that if $\Gamma \in [0, P]$, the level of conservatism of the robust solution can be adjusted through varying the value Γ to balance the robustness and quality of the solution.

Moreover, for obtaining a deterministic equation system, the strong duality theory of linear programming is used to transform (18) into (17) by introducing dual variables $p_{m,s}^{m'}$ and $X_{m,s}^{m'}$. Problem (18) is feasible and bounded for all $\Gamma_{m,s}^{m'} \in \left[0, |J_{m,s}^{m'}| \right]$, then the dual problem (17) is also feasible and bounded and their objective values coincide. Thus, we conclude that the function $\pi_1(\boldsymbol{\beta^*}, \Gamma_{m,s}^{m'})$ is equal to the objective function value of problem (17). □

Similarly, another set of constraints with uncertain parameters can be given as the following proposition.

Proposition 2. The protection function

$$\pi_2(\Gamma_{m,s}^{m'}) = \max_{\boldsymbol{S}} \left\{ \hat{e}_{m,s}^{m'} + (\Gamma_{m,s}^{m'} - \lfloor \Gamma_{m,s}^{m'} \rfloor) \hat{e}_{m,t}^{m'} \right\}, \tag{19}$$

in (15) is equivalent to the objective function of the following linear programming problem by introducing decision variables $z_{m,s}^{m'}$,

$$\begin{aligned} \pi_2(\Gamma_{m,s}^{m'}) = \max \quad & \hat{e}_{m,s}^{m'} z_{m,s}^{m'} \\ \text{s.t.} \quad & \begin{cases} z_{m,s}^{m'} \leq \Gamma_{m,s}^{m'}, & \forall (m, m', s) \in T, \\ 0 \leq z_{m,s}^{m'} \leq 1, & \forall (m, m', s) \in T, \end{cases} \end{aligned} \tag{20}$$

and the dual problem of (20) is formulated as (21) by introducing dual variables $q_{m,s}^{m'}$ and $V_{m,s}^{m'}$.

$$\begin{aligned} \min \quad & q_{m,s}^{m'} + \Gamma_{m,s}^{m'} V_{m,s}^{m'} \\ \text{s.t.} \quad & \begin{cases} V_{m,s}^{m'} + q_{m,s}^{m'} \geq \hat{e}_{m,s}^{m'}, & \forall (m, m', s) \in T, \\ q_{m,s}^{m'} \geq 0, & \forall (m, m', s) \in T, \\ V_{m,s}^{m'} \geq 0, & \forall (m, m', s) \in T. \end{cases} \end{aligned} \tag{21}$$

According to Propositions 1 and 2, the complete robust counterpart of (12) can be equivalently converted to the following linear formulation after some simplifications.

$$
\begin{aligned}
&\text{Min} \quad \sum_{(m,m',s)\in T}\sum_{i'\in N_{m'}}\sum_{i\in N_m} \rho_1 c_{m,s}^{m',i'}\left(L_{i,s}^m - A_{i',s}^{m'} - e_{m,s}^{m'}\right)\beta_{m,i,s}^{m',i'} \\
&+\sum_{m\in M}\sum_{i\in N_m}\sum_{s\in S_m}\rho_2\frac{b_{i,s}^m}{2}(L_{i,s}^m - L_{i-1,s}^m)^2 + \sum_{(m,m',s)\in T}(p_{m,s}^{m'} + \Gamma_{m,s}^{m'}X_{m,s}^{m'})
\end{aligned}
$$

$$
\text{s.t.}\begin{cases}
X_{m,s}^{m'} + p_{m,s}^{m'} \geq \sum_{i'\in N_{m'}}\sum_{i\in N_m}(\rho_1 c_{m,s}^{m',i'}\beta_{m,i,s}^{m',i'})\hat{e}_{m,s}^{m'} & \forall(m,m',s)\in T \\
p_{m,s}^{m'} \geq 0 & \forall(m,m',s)\in T \\
X_{m,s}^{m'} \geq 0 & \forall(m,m',s)\in T \\
M(\alpha_{m,i,s}^{m',i'} - 1) - L_{i,s}^m + A_{i',s}^{m'} + e_{m,s}^{m'} + q_{m,s}^{m'} + \Gamma_{m,s}^{m'}V_{m,s}^{m'} \leq 0 & \\
 & \forall(m,m',s)\in T, i\in N_m, i'\in N_{m'} \\
-M\alpha_{m,i,s}^{m',i'} + L_{i,s}^m - A_{i',s}^{m'} - e_{m,s}^{m'} + q_{m,s}^{m'} + \Gamma_{m,s}^{m'}V_{m,s}^{m'} \leq 0 & \\
 & \forall(m,m',s)\in T, i\in N_m, i'\in N_{m'} \\
V_{m,s}^{m'} + q_{m,s}^{m'} \geq \hat{e}_{m,s}^{m'} & \forall(m,m',s)\in T \\
q_{m,s}^{m'} \geq 0 & \forall(m,m',s)\in T \\
V_{m,s}^{m'} \geq 0 & \forall(m,m',s)\in T \\
\text{Constraints } (1)-(7),(9),(13). &
\end{cases} \tag{22}
$$

The above MINLP model (22) with binary variables is a computationally difficult problem to solve when there are numerous routes and transfer stations for large-scale cases. In order to alleviate the computational burden, a Benders decomposition based approach is proposed in the following section.

4. Generalized Benders decomposition based solution approach

The Benders decomposition algorithm was first proposed by Benders (1962), and it can be an efficient method to solve mixed integer linear programming (MILP) problems. Geoffrion (1972) generalized the decomposition technique to nonlinear problems. The generalized Benders decomposition (GBD) technique divides the original problem into two parts, which are the subproblem (SP) and the master problem (MP). With some variables, especially integer variables temporarily fixed to given values, the problem referred to as the SP can be easily solved.

The MP is formulated to optimize the integer variables with the other decision variables solved in the SP. The constraints in the MP include Benders cuts and original constraints that only involve the integer variables. The Benders cuts are added to the MP iteratively as new constraints. They comprise feasibility and optimality cuts constructed by the SP solutions. If the SP is infeasible, then a feasibility Benders cut is generated using the extreme rays. If the SP is solved optimally, an optimality Benders cut is generated with optimal multipliers. Thus, the solution of the integer variables to the MP is taken as new fixed values, and the SP is solved again. By repeating the above process and adding new Benders cuts, the feasible region of the MP becomes smaller and smaller. The algorithm is terminated when it converges to the optimal solution, which means the solution to the MP for integer variables is the same as the former iteration, or the difference between the MP and the SP reaches a precision requirement.

4.1. The subproblem (SP)

The discussed algorithm is applied to the robust counterpart model (22) for the timetable optimization problem. According to the definition and application of GBD, the binary decision variables $\alpha_{m,i,s}^{m',i'}$ defining whether two trains can connect with each other or not are the variables fixed in the SP. The other decision variables $A^m_{i,s}, L^m_{i,s}, R^m_{i,s}$, and $D^m_{i,s}$ $(m \in M, s \in S_m, i \in N_m)$ can be derived from the SP.

First, $\alpha^{m',i'}_{m,i,s}$ are assigned to arbitrary values $\bar{\alpha}^{m',i'}_{m,i,s}$, which are set the original connections between two trains according to the original timetable. While the fixed values may lead to the infeasibility of the SP, artificial variables $v^{m',i'}_{m,i,s}$ and $w^{m',i'}_{m,i,s}$ are introduced into the coupling constraints as well as the objective function. The SP is formulated as follows:

$$\begin{aligned}\text{Min} \quad & \sum_{(m,m',s)\in T} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \rho_1 c^{m',i'}_{m,s} \left(L^m_{i,s} - A^{m'}_{i',s} - e^{m'}_{m,s}\right) \beta^{m',i'}_{m,i,s} + M\left(v^{m',i'}_{m,i,s} + w^{m',i'}_{m,i,s}\right) \\ & + \sum_{m\in M} \sum_{i\in N_m} \sum_{s\in S_m} \rho_2 \frac{b^m_{i,s}}{2} (L^m_{i,s} - L^m_{i-1,s})^2 + \sum_{(m,m',s)\in T} (p^{m'}_{m,s} + \Gamma^{m'}_{m,s} X^{m'}_{m,s})\end{aligned}$$

$$\text{s.t.} \begin{cases} X^{m'}_{m,s} + p^{m'}_{m,s} \geq \sum_{i'\in N_{m'}} \sum_{i\in N_m} (\rho_1 c^{m',i'}_{m,s} \beta^{m',i'}_{m,i,s}) \hat{e}^{m'}_{m,s} : \boldsymbol{\gamma^{m'}_{m,s}}, & \forall (m,m',s)\in T, \\ p^{m'}_{m,s} \geq 0, & \forall (m,m',s)\in T, \\ X^{m'}_{m,s} \geq 0, & \forall (m,m',s)\in T, \\ M(\alpha^{m',i'}_{m,i,s} - 1) - L^m_{i,s} + A^{m'}_{i',s} + e^{m'}_{m,s} + q^{m'}_{m,s} + \Gamma^{m'}_{m,s} V^{m'}_{m,s} - v^{m',i'}_{m,i,s} \leq 0 : \boldsymbol{\delta^{m',i'}_{m,i,s}}, & \\ & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}, \\ -M\alpha^{m',i'}_{m,i,s} + L^m_{i,s} - A^{m'}_{i',s} - e^{m'}_{m,s} + q^{m'}_{m,s} + \Gamma^{m'}_{m,s} V^{m'}_{m,s} - w^{m',i'}_{m,i,s} \leq 0 : \boldsymbol{\eta^{m',i'}_{m,i,s}}, & \\ & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}, \\ V^{m'}_{m,s} + q^{m'}_{m,s} \geq \hat{e}^{m'}_{m,s}, & \forall (m,m',s)\in T, \\ q^{m'}_{m,s} \geq 0, & \forall (m,m',s)\in T, \\ V^{m'}_{m,s} \geq 0, & \forall (m,m',s)\in T, \\ v^{m',i'}_{m,i,s} \geq 0, & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}, \\ w^{m',i'}_{m,i,s} \geq 0, & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}, \\ \beta^{m',i'}_{m,i,s} = \alpha^{m',i'}_{m,i,s} - \alpha^{m',i'}_{m,i-1,s} : \boldsymbol{\zeta^{m',i'}_{m,i,s}}, & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}, \\ \alpha^{m',i'}_{m,i,s} = \bar{\alpha}^{m',i'}_{m,i,s} : \boldsymbol{\lambda^{m',i'}_{m,i,s}}, & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}, \\ \text{Constraints } (1)-(7), (13). & \end{cases} \tag{23}$$

In the above formulation, M is a large positive number and if $\bar{\alpha}^{m',i'}_{m,i,s}$ are feasible to the subproblem, it can be obtained that $v^{m',i'}_{m,i,s} = 0$ and $w^{m',i'}_{m,i,s} = 0$. With fixed values $\bar{\alpha}^{m',i'}_{m,i,s}$, we can attain a mixed integer nonlinear programming subproblem that is much easier to handle than the original mixed integer nonlinear programming problem. The constraints of the SP are linear and the optimal values of the dual variables can be obtained according to the application of the well-known KKT conditions. Let $\gamma^{m'}_{m,s}, \delta^{m',i'}_{m,i,s}, \eta^{m',i'}_{m,i,s}, \zeta^{m',i'}_{m,i,s}$, and $\lambda^{m',i'}_{m,i,s}$ be the dual variables assigned to the constraints in formulation (23) respectively, which are in bold in (23). They are used to establish a Lagrangian dual problem as dual multipliers. The domain of each dual variable depends on the property of the related constraint and for example $\lambda^{m',i'}_{m,i,s} \in \boldsymbol{R}$.

Applying the KKT conditions for the SP and considering the optimal values $\bar{A}^m_{i,s}$ and $\bar{L}^m_{i,s}$ solved by the

SP, the optimal dual variables above can be obtained by solving the following equations.

$$\begin{cases} \lambda_{m,i,s}^{m',i'} = M\left(\eta_{m,i,s}^{m',i'} - \delta_{m,i,s}^{m',i'}\right) + \zeta_{m,i,s}^{m',i'} - \zeta_{m,i+1,s}^{m',i'}, & \forall (m,m',s) \in T, i, i+1 \in N_m, i' \in N_{m'}, \\ \zeta_{m,i,s}^{m',i'} = \rho_1 c_{m,s}^{m',i'}\left(L_{i,s}^{m} - A_{i',s}^{m'} - e_{m,s}^{m'}\right) + \rho_1 c_{m,s}^{m',i'} \hat{e}_{m,s}^{m'} \gamma_{m,s}^{m'}, & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ \left[\sum_{i' \in N_{m'}} \sum_{i \in N_m} (\rho_1 c_{m,s}^{m',i'} \beta_{m,i,s}^{m',i'}) \hat{e}_{m,s}^{m'} - X_{m,s}^{m'} - p_{m,s}^{m'}\right] \gamma_{m,s}^{m'} = 0, & \forall (m,m',s) \in T, \\ \left[M(\alpha_{m,i,s}^{m',i'} - 1) - L_{i,s}^{m} + A_{i',s}^{m'} + e_{m,s}^{m'} + q_{m,s}^{m'} + \Gamma_{m,s}^{m'} V_{m,s}^{m'} - v_{m,i,s}^{m',i'}\right] \delta_{m,i,s}^{m',i'} = 0, & \\ & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ \left(-M\alpha_{m,i,s}^{m',i'} + L_{i,s}^{m} - A_{i',s}^{m'} - e_{m,s}^{m'} + q_{m,s}^{m'} + \Gamma_{m,s}^{m'} V_{m,s}^{m'} - w_{m,i,s}^{m',i'}\right) \eta_{m,i,s}^{m',i'} = 0, & \\ & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ \gamma_{m,s}^{m'},\ \delta_{m,i,s}^{m',i'},\ \eta_{m,i,s}^{m',i'} \geq 0, & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}, \\ \zeta_{m,i,s}^{m',i'},\ \lambda_{m,i,s}^{m',i'} \in \boldsymbol{R}, & \forall (m,m',s) \in T, i \in N_m, i' \in N_{m'}. \end{cases} \tag{24}$$

When the integer variables $\alpha_{m,i,s}^{m',i'}$ are set as fixed values $\bar{\alpha}_{m,i,s}^{m',i'}$, new constraints $\alpha_{m,i,s}^{m',i'} = \bar{\alpha}_{m,i,s}^{m',i'}$ are added into the optimization model (23). $\bar{\alpha}_{m,i,s}^{m',i'}$ are known constants, while $\alpha_{m,i,s}^{m',i'}$ and $\beta_{m,i,s}^{m',i'}$ are still variables. Thus, only the dual variables $\lambda_{m,i,s}^{m',i'}$ associated with the constraints $\alpha_{m,i,s}^{m',i'} = \bar{\alpha}_{m,i,s}^{m',i'}$ are used in the master problem of the GBD algorithm. Alternatively, if we directly replace the integer variables $\alpha_{m,i,s}^{m',i'}$ as the given constants $\bar{\alpha}_{m,i,s}^{m',i'}$ in the optimization model (23), all the dual variables involving the constraints with constants $\bar{\alpha}_{m,i,s}^{m',i'}$ need to be used in the master problem of the GBD algorithm. These two ways for determining the dual variables used in the master problem are equivalent.

The SP is a more restricted problem than the original problem, and the optimal value of its objective function value provides an upper bound of the original problem. Let the objective function values of the SP be written as $Z^k (k = 1, ..., K)$ for the kth iteration, K means the maximum iteration, and the upper bound UB^k is constructed as follows:

$$UB^k = Z^k, \qquad k = 1, \ldots, K. \tag{25}$$

4.2. The master problem (MP)

In the GBD algorithm, when some decision variables are temporarily fixed, there are three following conditions:

- The SP is feasible and has one bounded and optimal solution. Then an optimality cut should be added into the MP.
- The SP is feasible but unbounded. Then the algorithm stops due to the original problem becomes also unbounded (Geoffrion, 1972).
- The SP is infeasible. Then a feasibility cut should be added into the MP.

Due to the fixed values of some of the integer variables, the coupling constraint (8) may not be feasible. Given the optimal objective $\bar{Z}^k (k = 1, ..., K)$ of the SP and the corresponding optimal multipliers concerning the constraints in the SP, the feasibility and optimality cut functions are constructed as follows, respectively:

$$J_{fea}^{k}(\alpha_{m,i,s}^{m',i'}) = \bar{\delta}_{m,i,s}^{m',i'(k)} \left[M(\alpha_{m,i,s}^{m',i'} - 1) - \bar{L}_{i,s}^{m(k)} + \bar{A}_{i',s}^{m'(k)} + e_{m,s}^{m'} + \bar{q}_{m,s}^{m'(k)} + \Gamma_{m,s}^{m'} \bar{V}_{m,s}^{m'(k)}\right]$$
$$+ \bar{\eta}_{m,i,s}^{m',i'(k)} \left(-M\alpha_{m,i,s}^{m',i'} + \bar{L}_{i,s}^{m(k)} - \bar{A}_{i',s}^{m'(k)} - e_{m,s}^{m'} + \bar{q}_{m,s}^{m'(k)} + \Gamma_{m,s}^{m'} \bar{V}_{m,s}^{m'(k)}\right), \tag{26}$$

$$J_{op}^{k}(\alpha_{m,i,s}^{m',i'}) = \bar{Z}^k + \sum_{(m,m',s)\in T} \sum_{i'\in N_{m'}} \sum_{i\in N_m} \bar{\lambda}_{m,i,s}^{m',i'(k)} (\alpha_{m,i,s}^{m',i'} - \bar{\alpha}_{m,i,s}^{m',i'(k)}). \tag{27}$$

The Benders cuts along with the constraints of the original problem relating to $\alpha_{m,i,s}^{m',i'}$ construct the MP. The objective function of the MP is to minimize μ with decision variables $\alpha_{m,i,s}^{m',i'}$. At one iteration of the algorithm, if the SP is infeasible (i.e., artificial variables $v_{m,i,s}^{m',i'}$ and $w_{m,i,s}^{m',i'}$ in model (23) do not equal 0), a new feasibility cut $J_{fea}^{k}(\alpha_{m,i,s}^{m',i'}) \leq 0$ is added to the MP. If the SP can be solved to obtain an optimal solution (i.e., artificial variables $v_{m,i,s}^{m',i'}$ and $w_{m,i,s}^{m',i'}$ equal 0), a new optimality cut $J_{op}^{k}(\alpha_{m,i,s}^{m',i'}) \leq \mu$ is added to the MP. The number of feasibility cuts and optimality cuts increases with each iteration. Thus, the MP is constructed with K_f feasibility cuts and K_p optimality cuts as follows,

$$\text{Min} \quad \mu$$
$$\text{s.t.} \begin{cases} J_{fea}^{j}(\alpha_{m,i,s}^{m',i'}) \leq 0, & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}; j = 1,2,...,K_f, \\ J_{op}^{t}(\alpha_{m,i,s}^{m',i'}) \leq \mu, & \forall (m,m',s)\in T, i\in N_m, i'\in N_{m'}; t = 1,2,...,K_p, \\ \mu \geq 0. \end{cases} \tag{28}$$

where $K_f + K_p = K$. μ is a scalar and its bound can be given from practical considerations here.

It should be noted that the MP is a slack version of the original problem, and it approximates below from the objective function of the original problem. Thus, for the kth iteration, the optimal value of the objective function of the problem (28) is a lower bound of the optimal objective function value of the original problem, i.e.,

$$LB^k = \mu^k, \qquad k = 1, ..., K. \tag{29}$$

4.3. Solution Algorithm

We summarize the steps of the generalized Benders decomposition technique based algorithm for the train timetable coordination optimization problem as the following Algorithm 1.

For the train timetable coordination optimization problem of a large-scale metro network, the proposed algorithm can quickly decompose the problem into the MP that determines the connection between trains and the SP that generates timetables with arrival and departure times of each train. Note that the MP with increasing Benders cuts here is NP-hard. The size of the formulated MP is relatively small, even for a large-scale urban metro network. It can be efficiently solved using existing optimization tools, such as the CPLEX solver. Once the integer variables are determined, it can finally converge to the optimal solution within a finite number of iterations since the SP is a convex quadratic program, which can be solved fast using commercial solvers. Therefore, the proposed algorithm provides important and practical decision support for a computationally efficient train timetable generation of a large-scale metro network.

Based on the results proposed by Geoffrion (1972), we can straightforwardly obtain the following two propositions.

Proposition 3. The objective function of subproblem (23) is finite and the corresponding optimal dual multipliers can be found with fixed values $\bar{\alpha}_{m,i,s}^{m',i'}$.

Algorithm 1 Generalized Benders decomposition algorithm

Step 1: Initialization. Set the iteration counter $k = 1$ and a tolerance value ϵ, set the integer variables $\alpha_{m,i,s}^{m',i'}$ as initial values $\bar{\alpha}_{m,i,s}^{m',i'(k)}$.

Step 2: Solution to the Subproblem. If the SP is feasible with fixed $\bar{\alpha}_{m,i,s}^{m',i'(k)}$ according to (23), solve the SP to obtain the objective function value Z^k, decision variables $A_{i,s}^{m(k)}$ and $L_{i,s}^{m(k)}$ and dual variable values $\lambda_{m,i,s}^{m',i'(k)}, \delta_{m,i,s}^{m',i'(k)}, \eta_{m,i,s}^{m',i'(k)}$. Otherwise, if the subproblem is infeasible with fixed $\bar{\alpha}_{m,i,s}^{m',i'(k)}$, the solution to the SP can be obtained by solving the relaxed subproblem. Update UB^k.

Step 3: Generation of Benders cuts. With the solution to the SP, generate the feasibility cut $J_{fea}^k(\alpha_{m,i,s}^{m',i'}) \leq 0$ as Equation (26) or the optimality cut $J_{op}^k(\alpha_{m,i,s}^{m',i'}) \leq \mu$ as Equation (27), then add the cut into the MP in **Step 4**.

Step 4: Solution to the Master Problem. Solve the master problem (28) with all Benders cuts. Obtain μ^k and the new values of the integer variables $\bar{\alpha}_{m,i,s}^{m',i'(k+1)}$. Update LB^k.

Step 5: Convergence check. If $UB^k - LB^k \leq \epsilon$, the algorithm terminates. Otherwise, the iteration counter $k \leftarrow k + 1$, and the algorithm continues with **Step 2-5**.

Proposition 4. The solution technique converges to the optimal solution in a finite number of iterations with an optimality gap ϵ.

5. Numerical simulations

In this section, we conduct two numerical experiments to illustrate the developed robust model and the solution approach. Example 1 is a small metro network with 3 lines and 2 interchange stations used to demonstrate the effectiveness of the Benders decomposition algorithm procedure and the robustness of the proposed optimization model. Example 2 is used to validate the practicability of the developed robust model and the solution method to a real-world metro network based on operation data (e.g., headway, running time, dwell time restrictions, passenger demands) from the Beijing metro network. The proposed Benders decomposition algorithm is implemented using MATLAB R2018b on a PC (1.6-GHz processor speed and 16-GB memory size) with the platform of Windows 10, and the formulated subproblems are solved using YALMIP.

5.1. Example 1: a small metro network

To verify the effectiveness of the presented model and Benders decomposition algorithm for the timetable coordination problem, we consider a metro network with 3 lines, 2 transfer stations, and 11 normal stations, as shown in Figure 1. The interchange stations between lines 1, 2 and between lines 1, 3 are station $s1$ and $s2$ respectively. 10 trains are dispatched from the origin station of each line. According to the current schedules and standards from a similar network structure in the Beijing metro network, the allowable range of operational parameters and the nominal transfer walking times at two interchange stations are listed in Table 3. The weight coefficients in the objective function (22) are set as $\rho_1 = 1$ and $\rho_2 = 0.01$ which ensures the same magnitude order of the two terms. The arrival times of the first train at the origin station for all the lines are set to the same value. The numbers of passengers transferring among the three lines at interchange stations $s1$ and $s2$ are given in Table 4 during the transitional period. The average passenger arrival rates $b_{i,s}^m$ are time-dependent and different from lines and stations according to real-world data, and they are not displayed here due to their large number. The model for the small network includes 800 binary

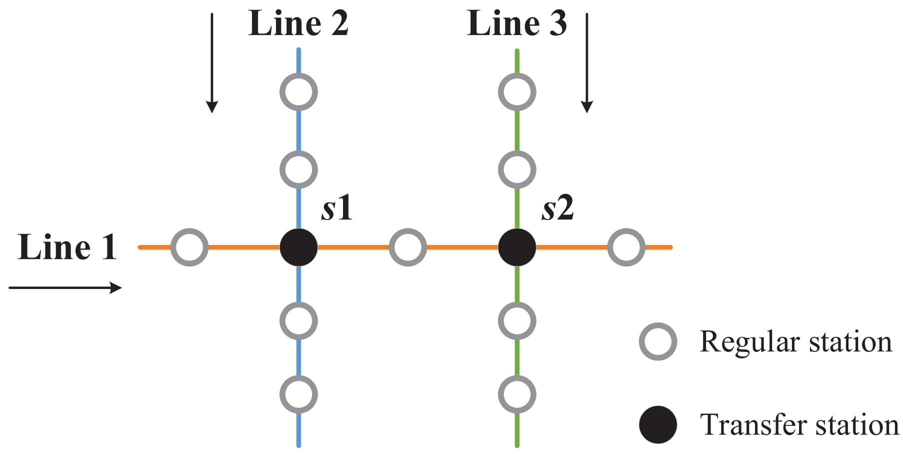

Figure 1: Illustration of a simple metro network.

Table 3: Operational parameters in the small metro network

Parameters	Range of standard values
Dwell time $D^m_{i,s}$	$[10, 60]s$
Running time $R^m_{i,s}$	$[120, 300]s$
Trip time $A^m_{i,S} - L^m_{i,1}$	$[600, 1500]s$
nominal transfer walking time $e^{m'}_{m,s}$	$e^1_{2,s1} = e^2_{1,s1} = 120s$;$e^1_{3,s2} = e^3_{1,s2} = 180s$

variables regarding $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and 613 continuous variables regarding $\boldsymbol{L}$, $\boldsymbol{A}$, $\boldsymbol{R}$, $\boldsymbol{D}$, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{V}$, $\boldsymbol{W}$. 701 equality constraints and 1667 inequality constraints are contained in the small case model.

Table 4: The number of passengers transferring at two interchange stations during transitional period (unit: person)

Station	Transfer	Train									
		1	2	3	4	5	6	7	8	9	10
$s1$	1-2	25	28	14	28	20	13	9	17	25	15
	2-1	2	15	15	8	13	8	7	14	12	9
$s2$	1-3	11	7	13	14	10	11	11	6	10	3
	3-1	18	11	11	8	9	28	23	13	28	11

5.1.1. Comparison among peak, transitional, off-peak periods

Three periods (i.e., the peak hours, the transitional period, and the off-peak hours) are considered by applying the deterministic model and proposed solution method. The time horizons and headway ranges for the three periods are given in Table 5. The objective function values (Q), including transfer waiting time ($Q1$) and access waiting time ($Q2$), are calculated with various numbers of transfer passengers $c^{m',i'}_{m,s}$ and different average passenger arrival rates $b^m_{i,s}$ corresponding to the three periods. The optimal solutions are compared with the original timetables with the same headway and dwell time for trains at the same station.

The total passenger waiting time is decreased by adjusting the arrival and departure time of each train slightly without influencing the number of connections between the two trains under the proposed method. The computational results in Table 5 show that the objective function value is improved by 10.68% during peak hours, by 12.99% during the transitional period, and by 15.56% during off-peak hours. It is efficient for our proposed deterministic model to coordinate the train timetables, reduce the transfer waiting time, and access passenger waiting time. As expected, since the headway is the largest, the greatest improvement on the objective function value is achieved during the off-peak period.

Table 5: Parameters and results of peak hours, transitional period and off-peak hours

Period	Time horizon	Headway range	$Q(s)$	$Q1(s)$	$Q2(s)$
		Line 1: [120,240]	$4.62{\cdot}10^4$	$3.72{\cdot}10^3$	$4.25{\cdot}10^6$
Peak	8:00-9:00	Line 2: [240,390]	Improvement	Improvement	Improvement
		Line 3: [180,300]	10.66%	0.27%	11.49%
		Line 1: [210,450]	$5.02{\cdot}10^4$	$7.68{\cdot}10^3$	$4.25{\cdot}10^6$
Transitional	9:00-11:00	Line 2: [300,570]	Improvement	Improvement	Improvement
		Line 3: [240,450]	12.99%	2.15%	15.39%
		Line 1: [420,600]	$9.98{\cdot}10^4$	$3.02{\cdot}10^4$	$6.96{\cdot}10^6$
Off-peak	11:00-13:00	Line 2: [540,720]	Improvement	Improvement	Improvement
		Line 3: [420,660]	15.56%	16.07%	13.78%

Next, we conducted experiments for the three periods by considering deterministic model and robust optimization model with $\Gamma = 4$ (i.e., $\Gamma^1_{2,s1} = \Gamma^2_{1,s1} = \Gamma^1_{3,s2} = \Gamma^3_{1,s2} = 1$) and 10% deviation of nominal transfer walking time respectively. Table 6 shows the results of the deterministic model (the objective value Q_d, transfer waiting time $Q1_d$ and access passenger waiting time $Q2_d$) and the robust model $(Q_r, Q1_r, Q2_r)$ and the relative differences between the results of the two models for peak, transitional and off-peak periods. It can be seen that the peak period is more sensitive than the other two periods to the disturbance of transfer walking time. When the transfer walking time during peak hours fluctuates in a certain range, the generated deterministic and robust timetables may be quite different from each other. It brings about a dramatic increase in the transfer waiting time ($Q1$).

Table 6: Effect of uncertain transfer walking time on three different periods

Period	$Q_d(s)$	$Q_r(s)$	$\frac{Q_r-Q_d}{Q_R}$	$Q1_d(s)$	$Q1_r(s)$	$\frac{Q1_r-Q1_d}{Q1_r}$	$Q2_d(s)$	$Q2_r(s)$	$\frac{Q2_r-Q2_d}{Q2_r}$
Peak	$4.62{\cdot}10^4$	$7.64{\cdot}10^4$	39.47%	$3.72{\cdot}10^3$	$3.69{\cdot}10^4$	89.92%	$4.25{\cdot}10^6$	$3.94{\cdot}10^6$	-7.76%
Transitional	$5.02{\cdot}10^4$	$6.45{\cdot}10^4$	22.19%	$7.68{\cdot}10^3$	$2.13{\cdot}10^4$	64.00%	$4.25{\cdot}10^6$	$4.32{\cdot}10^6$	1.54%
Off-peak	$9.98{\cdot}10^4$	$10.05{\cdot}10^4$	0.74%	$3.02{\cdot}10^4$	$3.59{\cdot}10^4$	15.91%	$6.96{\cdot}10^6$	$6.98{\cdot}10^6$	0.25%

5.1.2. Computation performance of GBD algorithm

To test the performance of the generalized Benders decomposition algorithm, we compare the solution by the proposed algorithm with a solution generated directly by the CPLEX solver. Using CPLEX 12.5, we obtained the optimal solution with a computational time of 1073s while the Benders decomposition algorithm

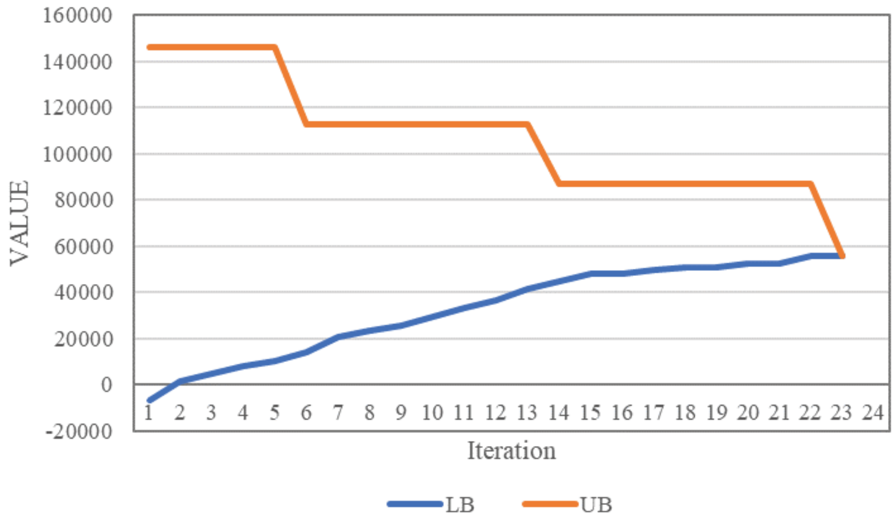

Figure 2: Evolution of the upper and lower bounds of the robust timetable coordination problem.

found the optimal solution in a relatively short computational time of 6.21s. The optimal objective value solved by both methods is $5.02 \cdot 10^4 s$, the total transfer waiting time is $7.68 \cdot 10^3 s$ and the total access passenger waiting time is $4.25 \cdot 10^6 s$. 4 optimality cuts and 19 feasibility cuts are added during the computational process for the small network. It can be inferred that in the face of large-scale metro network cases, the Benders decomposition technique has a more significant advantage than the CPLEX solver on the computation time. The evolution of the upper and lower bounds UB^k and LB^k is shown in Figure 2 and the algorithm is terminated when $UB^{23} = LB^{23}$. Figure 2 shows the convergence procedure of the proposed Benders decomposition algorithm, and the best bound is obtained at the 23rd iteration.

5.1.3. Influences of robust parameters and deviation range of uncertain transfer walking time

In order to further explore the advantage of the robust optimization method, we first calculated the optimal train timetables under three models respectively: the deterministic model with maximum transfer walking time (DMM), the deterministic model with average transfer walking time (DMA), and the robust optimization model (ROM) with uncertain transfer walking time. Then, with the calculated results, we compared the several indicators for these three models, including total objective value (Q), transfer waiting time ($Q1$), access passenger waiting time ($Q2$), and improvement between the robust model and other two deterministic models. 10 randomly generated cases with different realized transfer walking times are performed by taking 10% of the nominal transfer walking time as the maximum deviation range.

The comparison of the results in Table 7 shows that the ROM performs much better than the DMA. Under the DMA, an optimal timetable with more headway regularity is generated, which makes the access passenger waiting time of the DMA somewhat better than that of the ROM, as shown in Table 7. However, when the realized transfer walking time is larger than the average value among the 10 randomly generated cases, the transfer waiting time under the DMA will increase rapidly. It is because passengers may miss the train under the previous optimized timetable and have to wait for a longer time to board the next train. On the contrary, by incorporating the appropriate budget set of uncertain transfer walking time, the ROM can obtain a much better transfer waiting time than the DMA. In summary, as shown in Table 7, the mean value of the total objective (Q'') of the ROM is much fewer than that of the DMA (Q) by 51.66%. Hence, the ROM can achieve less passenger waiting time than the DMA under different realized transfer walking times.

Additionally, the DMM can generate an optimal timetable with relatively irregular and larger headways. It causes that the access transfer waiting time of the DMM is worse than that of the ROM, as shown in Table 7. On the other hand, in most cases, the DMM may bring about unnecessary and extra transfer waiting time due to the overestimation of the practical transfer walking time. In comparison, the ROM can obtain the timetable with less conservativeness than the DMM and achieve less transfer waiting time. As shown in Table 7, the mean value of the total objective (Q'') of the ROM is better than that of the DMM (Q') by 4.10%. Therefore, the ROM can achieve less passenger waiting time than the DMM in most cases, which shows that it is necessary to consider the uncertainty of the transfer walking time in our timetable scheduling model under metro networks.

Table 7: Comparison of the results under the different scenarios in the small network(unit: $\cdot 10^4 s$)

Models	Index	1	2	3	4	5	6	7	8	9	10	Mean
DMA	Q	9.97	7.54	9.95	7.50	12.47	10.30	12.36	12.34	12.54	12.86	10.783
	$Q1$	5.72	3.29	5.70	3.25	8.22	6.05	8.11	8.09	8.29	8.61	6.533
	$Q2$	4.25	4.25	4.25	4.25	4.25	4.25	4.25	4.25	4.25	4.25	4.25
DMM	Q'	5.51	5.58	5.49	5.53	5.41	5.41	5.29	5.28	5.49	5.37	5.436
	$Q1'$	1.19	1.26	1.17	1.21	1.09	1.09	0.97	0.96	1.17	1.05	1.116
	$Q2'$	4.32	4.32	4.32	4.32	4.32	4.32	4.32	4.32	4.32	4.32	4.32
ROM	Q''	5.20	5.27	5.19	5.23	5.10	5.52	4.99	4.97	5.18	5.48	5.213
	$Q1''$	0.92	0.99	0.91	0.95	0.82	1.24	0.71	0.69	0.90	1.20	0.933
	$Q2''$	4.28	4.28	4.28	4.28	4.28	4.28	4.28	4.28	4.28	4.28	4.28
Improve(%)	$\frac{(Q-Q'')}{Q}$	47.84	30.11	47.84	30.27	59.10	46.41	59.63	59.73	58.69	57.39	51.66
	$\frac{(Q'-Q'')}{Q'}$	5.62	5.56	5.46	5.42	5.73	-2.03	5.67	5.87	5.65	-2.05	4.10

Next, for the sake of exploring the influence of the robust parameters Γ on the objective values, transfer waiting time, and access passenger waiting time, we conduct a set of experiments with different Γ from 1 to 4 with fixed deviation range of uncertain transfer walking time of 5%, 10% and 15% of $e_{m,s}^{m'}$. The results are summarized in Table 8. $\Gamma = 4$ means the transfer walking times of the four transfer nodes (i.e. $e_{2,s1}^{1}$,$e_{1,s1}^{2}$, $e_{3,s2}^{1}$ and $e_{1,s2}^{3}$) are set as uncertain parameters. As is shown in Table 8, the objective value(Q) and transfer waiting time($Q1$) increase rapidly, and access passenger waiting time ($Q2$) increases gradually with larger values of Γ. When $\Gamma = 0$, the optimal results are consistent no matter the maximum deviation range of $e_{m,s}^{m'}$. It is the least conservative level of uncertainty of transfer walking time, and it makes the objective function value the smallest. On the contrary, the most conservative solution and the largest objective value are obtained when $\Gamma = 4$. Table 8 shows the tradeoff between the level of conservatism on the objective values. Also, $Q, Q1$ and $Q2$ increase with the growing deviation range and fixed Γ (except $\Gamma = 0$).

Table 8: The influence of the parameter Γ on results during peak period (unit: s)

Deviation	Index	$\Gamma = 0$	$\Gamma = 1$	$\Gamma = 2$	$\Gamma = 3$	$\Gamma = 4$
5%	Q	$5.02{\cdot}10^3$	$5.19{\cdot}10^4$	$5.39{\cdot}10^4$	$5.45{\cdot}10^4$	$5.73{\cdot}10^4$
	$Q1$	$7.68{\cdot}10^3$	$9.43{\cdot}10^3$	$1.13{\cdot}10^4$	$1.18{\cdot}10^4$	$1.45{\cdot}10^4$
	$Q2$	$4.25{\cdot}10^6$	$4.25{\cdot}10^6$	$4.27{\cdot}10^6$	$4.27{\cdot}10^6$	$4.28{\cdot}10^6$
10%	Q	$5.02{\cdot}10^3$	$5.36{\cdot}10^4$	$5.77{\cdot}10^4$	$5.89{\cdot}10^4$	$6.45{\cdot}10^4$
	$Q1$	$7.68{\cdot}10^3$	$1.12{\cdot}10^4$	$1.49{\cdot}10^4$	$1.60{\cdot}10^4$	$2.13{\cdot}10^4$
	$Q2$	$4.25{\cdot}10^6$	$4.25{\cdot}10^6$	$4.28{\cdot}10^6$	$4.29{\cdot}10^6$	$4.32{\cdot}10^6$
15%	Q	$5.02{\cdot}10^3$	$5.54{\cdot}10^4$	$6.15{\cdot}10^4$	$6.32{\cdot}10^4$	$7.25{\cdot}10^4$
	$Q1$	$7.68{\cdot}10^3$	$1.29{\cdot}10^4$	$1.85{\cdot}10^4$	$2.01{\cdot}10^4$	$2.82{\cdot}10^4$
	$Q2$	$4.25{\cdot}10^6$	$4.24{\cdot}10^6$	$4.30{\cdot}10^6$	$4.31{\cdot}10^6$	$4.44{\cdot}10^6$

Table 9 shows the transfer situation between two trains on connecting lines, in which the mark "-" represents a failed connection. It can be seen that all the passengers who need to transfer on the 10 trains at station $s1$ and $s2$ on line 1 transfer successfully to lines 2 and 3. All the trains connected successfully from line 3 to 1 at station $s2$, and only the first 9 trains at station $s1$ on line 2 could connect with trains on line 1. Figure 3 shows the time-distance diagrams at transfer station $s1$ between lines 1 and 2 and transfer station $s2$ between lines 1 and 3. It is noted that the passengers in the second train of line 2 could transfer successfully with the second train of line 1 after optimization instead of connecting with the first train before optimization. With the slight adjustment of arrival and departure times of each train, the total passenger waiting time can be reduced. We next enlarge the simple network example to bi-direction, which means there are 16 transfer nodes. As is shown in Figure 4, the objective values and transfer waiting time almost increase similarly with Γ, while the access passenger waiting time with the weight coefficients $\rho_2 = 2 \cdot 10^{-3}$ increases at a slower rate.

Table 9: The transfer situation between two connecting lines ($\Gamma = 4$, $\hat{e}_{m,s}^{m'} = 15s$)

Station	Transfer	Train									
		1	2	3	4	5	6	7	8	9	10
$s1$	1-2	1	2	3	4	5	6	7	8	9	10
	2-1	2	3	4	5	6	7	8	9	10	-
$s2$	1-3	1	2	3	4	5	6	7	8	9	10
	3-1	2	3	4	5	6	7	8	9	10	-

5.2. Example 2: a large-scale Beijing metro network

To demonstrate the practicability of the proposed model and solution method, we run some experiments on a large-scale metro network with a set of routes and stations. The network in Figure 5 is the Beijing metro network consisting of 15 operating lines and 51 transfer stations, where the operation direction is marked with arrows in Figure 5.

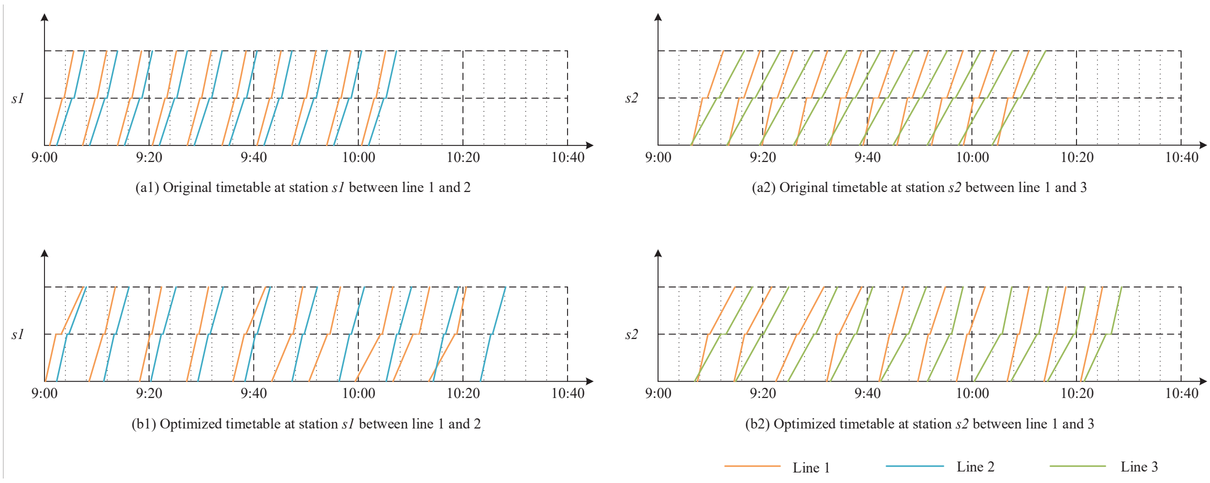

Figure 3: Time-distance diagrams at transfer stations $s1$ and $s2$.

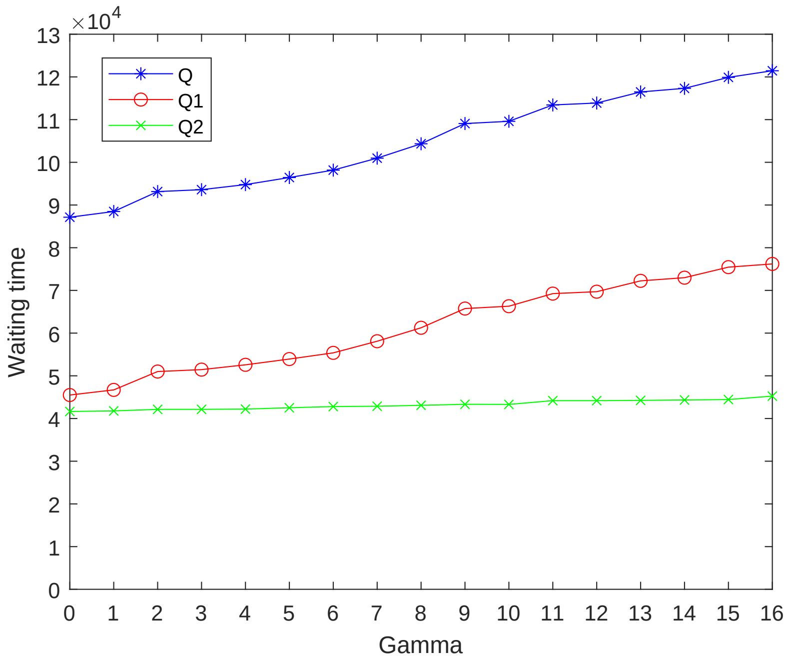

Figure 4: Optimal objective value with respect to different Γ.

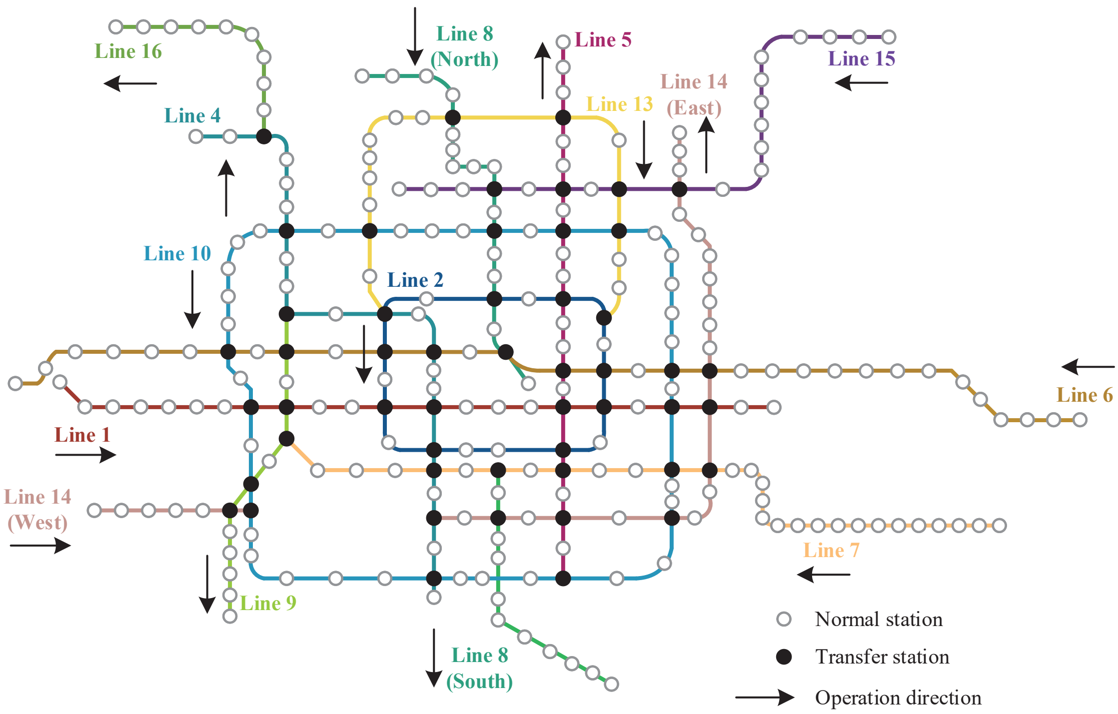

Figure 5: Illustration of part of Beijing metro network.

The planning horizon considered in this experiment is from 09:00 to 11:00, and 150 trains (10 trains for each line) are dispatched from the first station of each line. The minimum and maximum headway of each line are displayed in Table 10, and the other operational parameters are listed in Table 11. The weight coefficients are specified as $\rho_1 = 1$ and $\rho_2 = 2 \cdot 10^{-1}$ in the objective function. In addition, the transfer walking time $e_{m,s}^{m'}$ between connecting lines m and m' at interchange station s is set according to the practical operation data of the Beijing metro network. The maximum deviation $\hat{e}_{m,s}^{m'}$ is set as 10% of the nominal transfer walking time. The passenger demands are set according to the actual records of Beijing Metro Network on a workday, and they are not displayed here on account of the large number. Considering 95 interchange nodes, 15 involved lines, 10 dispatched trains, and the actual number of stations for each line, the model for the Beijing metro network contains 19000 binary variables, 15181 continuous variables, 15181 equality constraints, and 37401 inequality constraints.

Table 10: The minimum and maximum headway of each line for the experimented Beijing metro network (unit: s)

Line	1	2	4	5	6	7	8 North	8 South
Minimum	150	150	150	150	180	180	210	210
Maximum	300	300	300	300	360	360	390	390
Line	9	10	13	14 West	14 East	15	16	
Minimum	150	150	180	180	150	210	210	
Maximum	300	300	360	360	300	390	390	

Table 11: Operational parameters in the Beijing metro network

Parameters	Range of standard values
Station number	23,18,10,24,23,22,16,19,13,45,12,23,17,7,12
Dwell time $D^m_{i,s}$	$[10, 60]s$
Running time $R^m_{i,s}$	$[120, 300]s$
Trip time $A^m_{i,S} - L^m_{i,1}$	$[1000, 12000]s$

We first try to solve the robust model with parameter $\Gamma = 95$ using the CPLEX solver and obtain a solution with an objective function value of $1.31{\cdot}10^7s$ and 107.43% gap when using a computational time limit of $3600s$. Then with the Benders decomposition method and the preset 10% gap, the objective value of the robust model is found to be $4.87{\cdot}10^6s$ as shown in Table 12. The computational time of the Benders decomposition method was $395s$, and it is markedly smaller than that of the CPLEX solver. 6 optimality cuts and 27 feasibility cuts are generated in the computational process of the large-scale Beijing metro network case. Moreover, compared to the original timetable with the same headway for the trains at the same station, the objective value is reduced by 14.11%, and the most significant decrease is 20.15% for the total transfer waiting time after optimization. Table 13 exhibits the transfer waiting time and the corresponding improvements at some representative interchange stations in the metro network. The improvement of each transfer node is calculated as the difference on the total transfer waiting time of the 10 trains before and after optimization. For instance, the transfer waiting time for passengers getting off from line 1 and catching train 1 on line 2 successfully at FuxingMen Station is $375s$. Similarly, the transfer waiting time for passengers getting off from line 2 and catching train 1 on line 1 successfully at FuxingMen Station is $435s$. Then, the total transfer waiting time for the 10 trains of FuxingMen Station from line 1 to 2 (1-2) after optimization is $4642s$, and the original total transfer waiting time is $36442s$. Thus the improvement is $36442 - 4642 = 31800s$. The increase on the transfer waiting times of a few transfer nodes can reduce the total passenger transfer waiting time of the entire metro network. Due to the limitation of the departure time of each train at the first station on each line, six pairs of lines with six transfer nodes could not be connected practically during the planning horizon, and passengers in the first several trains on some lines may wait for an unreasonable long time. The complete results are displayed in Appendix A.

Table 12: Comparisons of the original and optimized results (unit: s)

	Objective	Transfer waiting time	Access passenger waiting time
Original	$5.67{\cdot}10^6s$	$2.68{\cdot}10^6s$	$1.50{\cdot}10^7s$
Optimized	$4.87{\cdot}10^6s$	$2.12{\cdot}10^6s$	$1.38{\cdot}10^7s$
Improvement	14.11%	20.15%	8.00%

Table 13: Transfer waiting time at representative transfer stations after optimization (unit: s)

Station	Connecting Line	Train 1	2	3	4	5	6	7	8	9	10	Optimized total waiting time	Original total waiting time	Improvement
FuxingMen	1-2	375	585	375	435	405	552	180	805	465	465	4642	36442	31800
	2-1	435	2461	1105	435	285	495	360	375	-	-	5951	19500	13549
JianguoMen	1-2	7300	180	1750	180	225	315	525	240	405	330	11450	22456	11006
	2-1	285	910	150	667	240	375	345	-	-	-	2972	16950	13978
XiDan	1-4	165	1850	210	300	1875	180	605	400	990	-	6575	1410	-5165
	4-1	150	90	195	904	90	375	210	-	-	-	2110	5977	3867
Military Museum	1-9	180	105	150	180	210	1750	150	90	270	105	3190	12580	9390
Station	9-1	595	120	840	150	120	165	180	-	-	-	2290	7616	5326
GongzhuFen	1-10	675	770	645	715	750	-	-	-	-	-	3555	8250	4695
	10-1	16275	5115	495	450	4050	360	300	390	300	255	38890	45150	6260
GuoMao	1-10	285	480	225	210	360	105	1100	-	-	-	2765	9912	7147
	10-1	270	150	180	360	225	435	405	405	435	-	2865	14190	11325
DawangLu	1-14East	105	528	90	165	195	-	-	-	-	-	1887	2940	1053
	14East-1	5320	2905	1435	120	165	75	570	105	180	135	11010	11640	630
XuanwuMen	2-4	160	180	120	75	45	90	210	-	-	-	880	804	-76
	4-2	980	180	90	105	105	120	90	90	45	75	1880	5184	3304

To demonstrate the robustness of the proposed approach with the uncertainties of the transfer walking time, we consider 10 scenarios with different transfer walking times by taking 10% of the nominal transfer walking time as the maximum deviation range generated randomly. The experimented transfer walking times for each scenario include 95 elements, which are not displayed here in detail. Table 14 exhibits the results, including the objective value (Q and Q'), transfer waiting time ($Q1$ and $Q1'$), and access passenger waiting time ($Q2$ and $Q2'$) under these 10 scenarios. For example, we first obtained a set of train connections $\boldsymbol{\alpha}$ from the robust solution when $\Gamma = 95$. The optimal objective value (Q), transfer waiting time ($Q1$), and access passenger waiting time ($Q2$) are $4864390s$, $2108442s$ and $13779744s$ under Scenario 1 with the transfer walking times generated randomly. While the results become $Q' = 4871941s$, $Q1' = 2115316s$, and $Q2' = 13783128s$ with the fixed train connection $\boldsymbol{\alpha}$ using the generated random transfer walking times. It can be seen that under different scenarios the maximum relative difference between the optimal results and results with $\Gamma = 95$ are only 0.23% ($\frac{Q'-Q}{Q}$), 0.36% ($\frac{Q1'-Q1}{Q1}$) and 0.36% ($\frac{Q2'-Q2}{Q2}$) respectively. Thus, the robust solutions (train connection $\boldsymbol{\alpha}$) within $\Gamma = 95$ generated by the proposed robust optimization model can cope with most scenarios with the deviation of uncertain transfer walking times at the interchange stations. The results with good performance can be obtained even when the transfer walking time significantly fluctuates. The small relative differences on the objective values under the various scenarios further indicate the robustness of the proposed method to the uncertainties on the transfer walking time.

Table 14: Comparison of the results under the different scenarios (unit: s)

Scenario	Q(Optimal)	$Q'(\Gamma = 95)$	$\frac{Q'-Q}{Q}$	$Q1$(Optimal)	$Q1'(\Gamma = 95)$	$\frac{Q1'-Q1}{Q1}$	$Q2$(Optimal)	$Q2'(\Gamma = 95)$	$\frac{Q2'-Q2}{Q2}$
1	4864390	4871941	0.16%	2108442	2115316	0.33%	13779744	13783128	0.02%
2	4859222	4866770	0.16%	2107120	2114732	0.36%	13760510	13760192	-0.00%
3	4864884	4870750	0.12%	2112437	2117713	0.25%	13762236	13765189	0.02%
4	4893958	4897688	0.08%	2099226	2101226	0.10%	13973662	13982312	0.06%
5	4846187	4850714	0.09%	2120134	2124523	0.21%	13630265	13630956	0.01%
6	4848837	4856298	0.15%	2120983	2128424	0.35%	13639272	13639371	0.00%
7	4839765	4850714	0.23%	2123484	2124523	0.05%	13581405	13630956	0.36%
8	4835453	4842631	0.15%	2123099	2129566	0.31%	13544386	13555837	0.08%
9	4828219	4834441	0.13%	2126703	2130597	0.18%	13525601	13524379	-0.01%
10	4828925	4832603	0.08%	2126576	2131464	0.23%	13511114	13510031	-0.01%

6. Conclusion

This paper developed a computationally efficient train timetable generation method for large-scale urban metro networks. A mixed integer nonlinear programming model was presented to optimize the timetable by determining the arrival and departure time of each train in the metro network within the planning horizon. A robust optimization model was developed to handle uncertainty in the transfer walking time. Due to the proposed model consisting of a number of binary variables and associated constraints, computational complexity tends to be a significant problem for large-scale networks. Thus, a generalized Benders decomposition technique based approach was designed to cope with the robust optimization model. Two series of numerical examples were conducted to validate the performance and availability of the proposed model and approach.

The computational results showed that the generalized Benders decomposition technique based approach has a significant advantage in solving the large-scale metro network problem in a relatively short amount of computation time. The proposed model and method can generate a robust train timetable with as few transfer waiting times and access passenger waiting times as possible, concerning the uncertain transfer walking time. In addition, by adjusting the conservative robustness level, we find that the lower robustness conservative level of uncertain transfer walking time will bring a smaller objective function value.

Further research will focus on the following two aspects. On the one hand, in order to describe the over-saturated condition during peak periods more accurately, we can regard the number of transfer passengers, the number of passengers waiting at stations, the train load, and the passenger flow control as decision variables to construct the dynamic evolution equation of passenger flow. This will make the optimal timetable design problem for large-scale metro networks more complicated due to the nonlinear objective function and coupling constraints. On the other hand, to realize the simultaneous optimization of metro network timetable and passenger choices of travel paths more rigorously, we need to construct further the complex coupling constraints between the dynamic passenger choice behavior and the train timetable. We may incorporate other decomposition methods for the design of an efficient train coordination optimization algorithm, which will be another future research work.

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Appendix A. Results of large scale network problem using the Benders decomposition method

Table 15: Waiting time at all transfer stations after optimization (unit: s).

Station	Connecting Line	Train 1	2	3	4	5	6	7	8	9	10	Optimized	Original	Improvement
FuxingMen	1-2	375	585	375	435	405	552	180	805	465	465	4642	36442	31800
	2-1	435	2461	1105	435	285	495	360	375	-	-	5951	19500	13549
JianguoMen	1-2	7300	180	1750	180	225	315	525	240	405	330	11450	22456	11006
	2-1	285	910	150	667	240	375	345	-	-	-	2972	16950	13978
XiDan	1-4	165	1850	210	300	1875	180	605	400	990	-	6575	1410	-5165
	4-1	150	90	195	1000	90	375	210	-	-	-	2110	5977	3867
DongDan	1-5	120	75	150	285	195	210	666	970	-	-	2671	4785	2114
	5-1	-	-	-	-	-	-	-	-	-	-	0	0	0
Military Museum Station	1-9	180	105	150	180	210	1750	150	90	270	105	3190	12580	9390
	9-1	595	120	960	150	120	165	180	-	-	-	2290	7616	5326

GongzhuFen	1-10	675	770	645	715	750	-	-	-	-	-	3555	8250	4695
	10-1	16275	5115	5495	5450	4950	360	300	390	300	255	38890	45150	6260
GuoMao	1-10	285	480	225	210	360	105	1100	-	-	-	2765	9912	7147
	10-1	270	150	180	360	225	435	405	405	435	-	2865	14190	11325
DawangLu	1-14East	105	1332	90	165	195	-	-	-	-	-	1887	2940	1053
	14East-1	5320	2905	1435	120	165	75	570	105	180	135	11010	11640	630
XiZhiMen	2-4	19432	19584	18648	22878	12936	10512	1794	240	1250	480	107754	131667	23913
	4-2	195	-	-	-	-	-	-	-	-	-	195	0	-195
	2-13	2835	120	240	352	532	624	290	575	760	2016	8344	0	-8344
	4-13	240	545	-	-	-	-	-	-	-	-	785	0	-785
XuanwuMen	2-4	160	180	120	75	45	90	210	-	-	-	880	804	-76
	4-2	980	180	90	105	105	120	90	90	45	75	1880	5184	3304
ChongwenMen	2-5	135	650	174	135	875	90	60	-	-	-	2119	6068	3949
	5-2	1295	280	625	150	45	485	750	75	120	90	3915	9450	5535
YongheGong	2-5	180	165	459	270	1875	225	150	-	-	-	3324	8800	5476
	5-2	1683	195	1375	270	155	440	165	90	135	150	4658	12550	7892
ChegongZhuang	2-6	20265	13300	15525	15775	3125	405	870	4950	450	1219	75884	84498	8614
	6-2	24035	12330	5418	1705	90	2460	270	330	1007	555	48200	42350	-5850
ChaoyangMen	2-6	120	975	2900	165	864	-	-	-	-	-	5024	4647	-377
	6-2	345	580	-	-	-	-	-	-	-	-	920	644	-276
GulouDaJie	2-8North	270	150	225	2000	345	510	180	315	2000	195	6190	11032	4842
	8North-2	240	1190	360	375	240	385	180	-	-	-	2970	8928	5958
DongzhiMen	13-2	30	90	-	-	-	-	-	-	-	-	120	0	-120
PinganLi	4-6	1911	2385	285	315	1403	3792	270	315	135	240	11051	14924	3873
	6-4	225	3861	1494	270	567	300	120	-	-	-	5541	9115	3574
CaiShiKou	4-7	8879	3624	1526	45	875	135	135	75	75	210	15579	18390	2811
	7-4	135	120	195	60	75	-	-	-	-	-	585	2428	1843
National Library	4-9	-	-	-	-	-	-	-	-	-	-	0	0	0
JiaoMen West	4-10	5950	3125	2625	480	450	420	555	360	285	1530	15780	33840	18060
	10-4	390	375	360	375	360	2625	435	495	330	-	5745	24012	18267
HaiDianHuang-	4-10	875	105	75	325	150	90	105	135	60	90	2010	8400	6390
Zhuang	10-4	165	270	170	30	45	165	150	-	-	-	995	4545	3550
Beijingnan Railway Station	4-14East	120	280	150	750	120	375	255	-	-	-	2050	0	-2050
XiYuan	4-16	105	75	1500	135	-	-	-	-	-	-	1815	0	-1815
DongSi	5-6	105	180	1029	135	1232	75	175	1500	75	-	4506	5953	1447
	6-5	1772	332	165	135	468	195	165	90	1150	-	4472	3288	-1184
CiQiKou	5-7	275	90	1375	135	165	165	120	105	715	75	3220	9180	5960
	7-5	150	180	150	195	135	270	60	110	30	-	1280	5618	4338
SongjiaZhuang	10-5	195	2530	5805	450	315	405	-	-	-	-	9700	0	-9700
HuiXinXiJie-	5-10	1375	330	360	3250	255	1330	270	1260	2320	1495	12245	13192	947
NanKou	10-5	2175	385	875	165	1815	180	255	210	-	-	6060	9894	3834
LiShuiQiao	5-13	120	105	105	480	150	180	-	-	-	-	1140	528	-612
	13-5	4690	2800	1680	60	90	480	60	75	90	120	10145	10781	636
PuHuangYu	5-14East	150	90	45	90	315	250	135	45	90	135	1345	4718	3373
	14East-5	160	45	120	150	225	45	60	60	105	-	970	4680	3710
DaTunLu East	5-15	875	180	165	135	1575	150	1325	60	30	-	4495	5213	718
	15-5	170	920	150	510	200	90	165	-	-	-	2205	3480	1275
Nan Lu-ogu	6-8North	28275	38115	23280	18255	13936	14175	3685	2790	255	165	142931	152426	9495
Xiang	8North-6	1105	-	-	-	-	-	-	-	-	-	1105	0	-1105
BaiShiQiao	6-9	-	-	-	-	-	-	-	-	-	-	0	0	0
South	9-6	22200	28800	12960	9835	14100	3660	5240	4970	1485	195	103385	201250	97865
Cishou Si	6-10	-	-	-	-	-	-	-	-	-	-	0	0	0
	10-6	105525	117900	91875	93085	75160	66430	67960	79920	86635	78970	863460	918520	55060
Hujia Lou	6-10	14220	10725	6324	4355	620	2170	75	1064	180	675	40408	33840	-6568
	10-6	240	1332	2184	165	1025	105	-	-	-	-	5051	4752	-299
JintaiLu	6-14East	3675	2250	595	620	105	60	135	135	255	210	8040	8245	205
	14East-6	120	75	45	60	902	606	90	-	-	-	1898	5523	3625
Zhushi Kou	7-8South	210	30	150	-	-	-	-	-	-	-	390	0	-390
Beijing West Railway Station	7-9	-	-	-	-	-	-	-	-	-	-	0	0	0
Shuang Jing	7-10	13167	8625	3705	600	495	600	735	765	150	105	28947	29976	1029
	10-7	345	1755	3720	435	375	-	-	-	-	-	6630	9660	3030
Jiulong Shan	7-14East	2035	310	135	620	15	180	60	896	165	570	4986	8037	3051
	14East-7	90	105	150	1169	1272	75	105	-	-	-	2966	1320	-1646
BeituCheng	8North-10	9350	315	360	435	475	540	495	405	555	-	12930	26610	13680
	10-8North	725	925	585	980	4625	330	585	500	3335	-	12590	26190	13600
HuoYing	8North-13	4680	5200	1540	30	140	250	120	135	135	60	12290	13756	1466
	13-8North	240	45	630	90	120	1950	-	-	-	-	3075	4170	1095
Olympic Green	8North-15	3135	50	30	45	225	150	60	150	165	330	4340	7140	2800
	15-8North	165	150	30	1575	75	120	90	-	-	-	2205	2236	31
Liuli Qiao	9-10	3645	1210	-	-	-	-	-	-	-	-	4855	7350	2495
	10-9	28200	21875	11305	11160	10080	3465	360	1190	4455	330	92420	88590	-3830
Qili Zhuang	14West-9	30	90	240	60	280	1085	180	90	45	-	2100	0	-2100
Xiju	14West-10	7750	510	690	1170	330	285	1350	345	285	-	315	0	-315
ShaoyaoJu	10-13	10872	555	390	540	540	585	570	450	525	2835	17862	7404	-10458
	13-10	47190	43260	29810	22040	21505	17270	11495	11310	3724	580	208184	291360	83176
Zhichun Lu	10-13	-	-	-	-	-	-	-	-	-	-	0	0	0
	13-10	120	120	120	150	1980	105	630	-	-	-	3225	6360	3135
Shili He	10-14East	450	405	360	315	2375	360	855	1495	-	-	6615	15540	8925

	14East-10	7750	510	690	1170	330	285	1350	345	285	-	12715	16574	3859
Yongding	8South-14East	435	330	330	360	510	435	360	330	-	-	3090	15133	12043
MenWai	14East-8South	195	255	90	1240	195	210	195	180	270	165	2995	14040	11045
WangJing	14East-15	150	90	90	1085	45	60	165	3280	30	-	4995	6201	1206
	15-14East	55	180	135	195	30	195	462	1480	90	-	2822	5955	3133
WangJing	13-15	850	1025	75	105	875	90	135	180	-	-	3335	6360	3025
West	15-13	1880	45	15	120	105	120	105	75	60	135	2660	4333	1673

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