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Fluid Mechanics and Complex Variable Theory: Getting Past the 19th Century

Paul K. Newton

Abstract: The subject of fluid mechanics is a rich, vibrant, and rapidly developing branch of applied mathematics. Historically, it has developed hand-in-hand with the elegant subject of complex variable theory. The Westmont College NSF-sponsored workshop on the revitalization of complex variable theory in the undergraduate curriculum focused partly on identifying scientific areas where complex variables plays an active role. The author presented a lecture on this topic and here we highlight parts of that presentation and follow-up discussion.

Keywords: Complex variables, fluid mechanics, potential flow, Stokes flow, point vortex

1. INTRODUCTION

Fluid mechanics, a venerable and classic branch of theoretical mechanics, is also an integral part of modern applied mathematics, with complex variable theory playing a distinguished role. The challenging starting point of the subject are the partial differential equations that govern fluid flow, the Navier–Stokes equations:

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{1}{Re} \Delta \vec{u}, \tag{1}$$

$$\nabla \cdot \vec{u} = 0, \tag{2}$$

$$\vec{u}(\vec{x}, 0) = \vec{f}(\vec{x}). \tag{3}$$

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Here, $\vec{u}$ is the fluid velocity field, ∇p is the pressure gradient, and Re is the Reynolds number. These equations are written in dimensionless form, and the Reynolds number is defined as $Re \equiv \frac{UL}{\nu}$, where U is a characteristic velocity scale, L is a characteristic length scale, and ν is the fluid viscosity. The two main terms of equation (1) that make the subject of incompressible fluid mechanics interesting are (i) the nonlinear inertial term $\vec{u} \cdot \nabla \vec{u}$, and (ii) the pressure gradient term ∇p, acting simultaneously in a dynamical setting, with initial conditions for the velocity field (3), and additional boundary conditions if one is modeling fluid–solid interactions. The flow is incompressible (2). In the simplest of settings (low Re), the equations are capable of producing streamline patterns, like those down the left column of Figure 1 showing (potential) flow past an airfoil [11]. However, as the Reynolds number increases, the same configuration (right column of Figure 1) produces a much more complex and turbulent velocity field behind the airfoil. Equations (1) to (3) capture all of these features of the flow, both instantaneously and as time progresses.
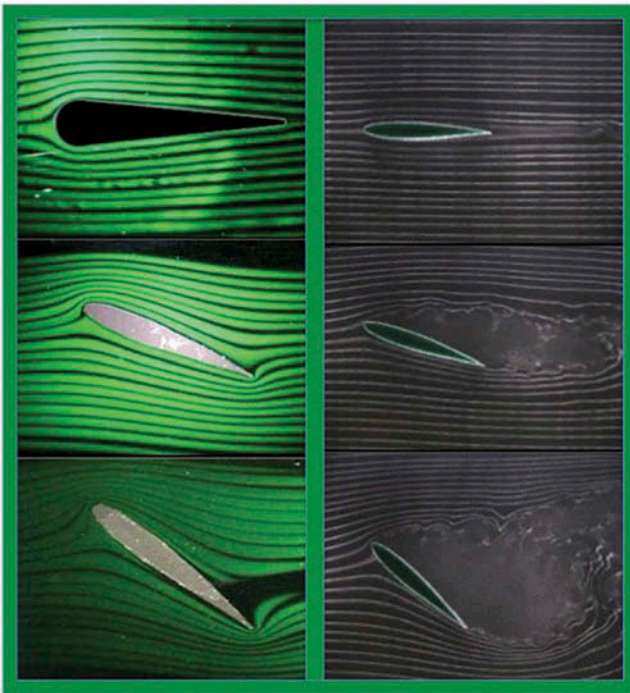

Figure 1. Left column: Low Reynolds number flow past an airfoil at increasing angle of attack. The flowfield produces streamline patterns that are closely approximated by potential flow approximations. Right column: High Reynolds number flow past an airfoil at increasing angle of attack. The flowfield produces a turbulent unsteady wake behind the airfoil. (web.mit.edu 13.021 - Marine Hydrodynamics Lecture 24C - Lifting Surfaces.)

Although much is known (mathematically) about the Navier–Stokes equations, there remain several nagging and fundamental open questions regarding the existence and smoothness of solutions and their relation to turbulent flow in the limit $Re \to \infty$. This is, in fact, the focus of one of the Clay Institute's Millenium problems (http://www.claymath.org/millennium-problems). At the heart of what makes fluid mechanics a challenging problem is the subtle interplay between the unsteady dynamics generated from initial and boundary conditions, and the non-linear term $\vec{u} \cdot \nabla \vec{u}$, which renders classic linear methods useless. Even in a nominally non-turbulent setting, such as the flow in an unsteady wake generated by insects, bats, hummingbirds, or fish [5], the relationship between the dynamical properties of the wake and the motion of the wing or tail that generated it, and how that influences maneuverability, is an exciting and rapidly developing field. One can get a glimpse of recent developments in this field by reading the collection of articles in the special issue of the *Journal of Nonlinear Science* entitled "Emergent Collective Behavior: From Fish Schools to Bacterial Colonies" [10].

An important tool used by many practitioners of fluid mechanics are complex variable methods which play a role in potential flow applications, boundary-integral formulations, wake-vortex models, and the analysis of small scale (e.g., biological) flows [6-9]. The two limiting regimes where complex variable theory can be used (in two-dimensions) are the Stokes limit $Re << 1$ governing small-scale (biological) flows, and very slow-moving flows associated with glaciers or lava, and the Euler limit $Re \to \infty$ governing large-scale flows with high velocities, such as most geophysical flows. The Stokes limit is particulary well-studied, and is governed by the linear biharmonic system:

$$\Delta \vec{u} = \nabla p, \tag{4}$$

$$\Delta^2 \vec{u} = 0. \tag{5}$$

The good news is that the governing equations are linear, but this can be misleading. Consider a squirming sperm cell (or collections of them) surrounded by incompressible fluid as shown in Figure 2. The moving solid (or semi-solid) boundary associated with the outer membrane of each cell is used by the sperm to "push-off" the fluid, exploiting the viscous boundary at the interface. This imparts a distributed force on the surrounding fluid, which allows the sperm cell to locomote. If one idealizes the situation, as one in which the sperm supplies it own motion, then it produces a velocity field in the surrounding fluid, much in the way a moving oar sets up motion in the fluid that surrounds it. However, an oar is inanimate and non-deformable, whereas a sperm cell typically senses the pressure the moving fluid field imposes back on its membrane, and adjusts its motion accordingly. Thus, any self-consistent model needs to take into account not only the equations (4) and (5), but also the embedded and fully coupled moving, deformable body governed by Newton's equations $F = ma$. In addition, other physical effects, such as chemical gradient fields released by each cell and

Figure 2. Human sperm fixed and stained with Mitotraches green to show mitochondria (and tail) and propidia iodide to show the nucleus. The fluid velocity field generated by the swimmers is governed by the linear Stokes equations (4) and (5), with viscous boundary conditions on the outer surface of each cell which allows the swimmers to locomote in the fluid. Cover image *Reproduction: The Journal of the Society for Reproduction and Fertility*, June 2004; 127(6) (Courtesy of Dr. David Becker, Dept. of Anatomy and Developmental Biology, UCL). © David Becker. Reproduced by permission of David Becker. Permission to reuse must be obtained from the rightsholder.

advected by the flow, may be part of the model. Thus, although we can be grateful for the linearity of the Stokes equations (4) and (5), there are still many nonlinear issues associated with coupled fluid-deformable systems and interactions among them that are at the forefront of current research [10].

In contrast, the Euler limit ($Re \to \infty$) is governed by the equations:

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = -\nabla p, \tag{6}$$

$$\nabla \cdot \vec{u} = 0. \tag{7}$$

In special circumstances, these equations give rise to potential flows [6-9, 11, 12, 16]. The potential flow equations (here we focus on two-dimensional systems) arise directly from writing the velocity field as the gradient of a potential:

$$\vec{u} = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = \nabla^{\perp} \psi = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right). \tag{8}$$

Here, $\phi(x,y)$ is referred to as the velocity potential, and $\psi(x,y)$ is called the streamfunction. This leads to the potential flow equation:

$$\Delta \phi = 0, \tag{9}$$

giving the most immediate and direct link to the theory of complex variables. However, from a dynamics point of view, there is more. Since:

$$\vec{u} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right), \tag{10}$$

the equations of motion for a fluid particle located at $(x(t), y(t))$ are Hamilton's canonical equations with the streamfunction playing the role of the Hamiltonian $\psi(x,y;t) = H(x,y;t)$ [13-15]. Since time-dependent Hamiltonians can produce chaotic trajectories, relatively simple but time-dependent flows can generate chaotic particle paths, which is the key to the interesting field of chaotic advection [4], a rapidly developing and active sub-field of fluid mechanics.

To highlight both the beauty and the limitations of potential flow, consider the following example of two-dimensional fluid flow of speed U_0 past a cylinder of radius a centered at $z = 0$.

Example 1. The complex potential

$$w(z) = \phi + i\psi = U_0(z + \frac{a^2}{z})$$

is one that gives rise to potential flow streamlines past a cylinder. The complex velocity, if set to zero, gives the stagnation points of the flow:

$$\frac{dw}{dz} = U_0(1 - \frac{a^2}{z^2}) = 0 \tag{11}$$

$$\Rightarrow z^2 = a^2, \quad x = \pm a; y = 0. \tag{12}$$

One can go further with this example and introduce *Blasius' Theorem* which relates the lift and drag around an object to the contour integral around the object.

Blasius Theorem:

$$F_x - iF_y = drag - ilift = \frac{i\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 dz.$$

For flow past a cylinder, we have:

$$\left(\frac{dw}{dz}\right)^2 = U_0^2 \left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right), \tag{13}$$

and since there is no $1/z$ term, there is no residue. This then implies:

$$\frac{i\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 dz = 0. \tag{14}$$

We can conclude that potential flow around a cylinder produces no lift, and no drag. This was known in the literature as d'Alembert's Paradox, after its obvious shortcomings to real flows was elucidated by d'Alembert in 1752. The resolution of this paradoxical result was not fully worked out until Ludwig Prandtl developed boundary layer theory in the early 20th century [1], when it was realized how viscous boundary conditions on the contour produce drag and lift by stopping the fluid at the boundary, which forces flow to rotate and shed vorticity into the wake [11]. From that point forward, most scientists focused on questions associated with the fully nonlinear time-dependent processes associated with the "shed vorticity" in the wake, the production of turbulence, and its quantitative description.

So, how to best convey both the beauty and the limitations of complex variable techniques to students of mathematics, engineering, and physics? Although complex variable theory is still widely used in potential flow applications ranging from Hele–Shaw flows, dendritic flows, free-boundary water wave problems, and certain porous media flows, all of these would require significant background material if introduced in a standard complex variable theory class populated with undergraduate mathematics majors. Complex variable methods also play a pivotal role in formulating many numerical methods that start with the boundary-integral formulation of solid–fluid interactions. However, this too would require some sophisticated background knowledge of numerical analysis. Instead, we focus on introducing some simple examples associated with vortex dynamics [12-16] applications where complex variable methods are also heavily used. With relatively little fluid mechanics background, this topic quickly introduces two aspects of fluid mechanics that are true 21st-century challenges and part of the Euler equations: dynamics (i.e., unsteady flows), and nonlinearity.

2. COMPLEX VARIABLE THEORY AND THE N-VORTEX PROBLEM

If there is vorticity in the flowfield, such as that shown in Figures 3 and 4, the problem becomes more interesting than if the flow is irrotational. Vorticity is defined as $\vec{\omega} = curl(\vec{u})$. Then, instead of equation (9), one has:

$$\Delta \psi = -\omega. \tag{15}$$

In the simplest context, one can assume that vorticity is concentrated at a single point, hence $\omega = \delta(r)$, where δ is the Dirac delta function, and the vortex is called a "point-vortex" [13]. It is via the point-vortex representation of the vorticity field that many research-level questions can be introduced that rely on complex variable techniques.

In polar coordinates, the velocity field of a point-vortex (u_r, u_θ), is given by:

$$u_r = 0, \tag{16}$$

$$u_\theta = \Gamma/2\pi r. \tag{17}$$

Here, $\Gamma \in R$ represents the strength of the point-vortex. Letting $z(t) = x(t) + iy(t)$, the velocity field can be written as an ordinary differential equation:

$$\dot{z}^* = \frac{\Gamma}{2\pi i} \frac{1}{z}. \tag{18}$$

Figure 3. An image of Hurricane Isabel as seen from the International Space Station (https://en.wikipedia.org/wiki/Hurricane_Isabel). An idealized hurricane produces a two-dimensional velocity field (vertical direction is compressed) approximating a point-vortex flow, radially decaying from the central eye with distance like $1/r$.
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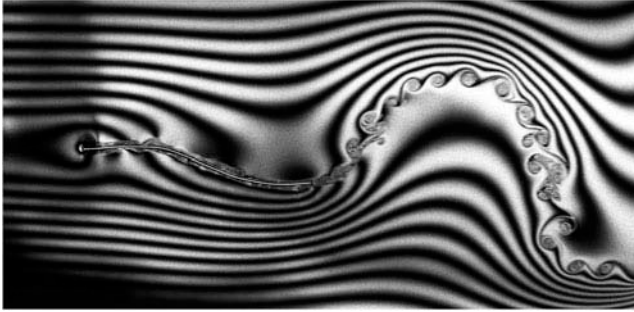
Figure 4. Typical organized vortex wake behind a blunt-body such as a cylinder, bat wing, or fish tail. Once the vortices are created by the object, they move under each other's influence (approximately) as a collection of point-vortices. Two-dimensional soap films offer a convenient way of creating and visualizing the vortex street behind and object, as shown in Figure 2 of [17]. Courtesy of Dr. Jun Zhang, NYU Department of Physics (http://www.physics.nyu.edu/jz11/). © Jun Zhang. Reproduced by permission of Jun Zhang. Permission to reuse must be obtained from the rightsholder.

To see this, let $z = r(t)\exp(i\theta(t))$. Then $\dot{z} = \dot{r}\exp(i\theta) + i\dot{\theta}r\exp(i\theta)$, and:

$$\dot{z}^* = \dot{r}\exp(-i\theta) - i\dot{\theta}r\exp(-i\theta) = \frac{\Gamma}{2\pi i}\frac{1}{r}\exp(-i\theta). \qquad (19)$$

When separated into real and imaginary parts, this yields:

$$\dot{r} = 0 = u_r, \qquad (20)$$

$$r\dot{\theta} = \frac{\Gamma}{2\pi r} = u_\theta, \qquad (21)$$

confirming equations (16) and (17). Based on the representation shown in equation (18), we now develop the N-vortex equations of motion. First, we let the singularity be located at the arbitrary point z_β in the complex plane, and have arbitrary strength $\Gamma_\beta \in R$:

$$\dot{z}^* = \frac{\Gamma_\beta}{2\pi i}\frac{1}{z - z_\beta}. \qquad (22)$$

By linear superposition, the field produced by N point-vortices, $\beta = 1, ..., N$ then becomes:

$$\dot{z}^* = \frac{1}{2\pi i}\sum_{\beta=1}^{N}\frac{\Gamma_\beta}{z - z_\beta}. \qquad (23)$$

Finally, we make the assumption (originally due to Helmholtz) that each of the point-vortices moves according to the local velocity field, hence:

$$\dot{z}_\alpha^* = \frac{1}{2\pi i} \sum_{\beta=1}^{N} {}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{24}$$

This is the classic dynamical system typically called the N-vortex problem [12–15]. The $'$ on the summation tells us that $\beta \neq \alpha$. The beauty of this efficient representation of the flow is that it compactly incorporates the key ingredients that make the Euler equations (6) interesting: nonlinearity and dynamics. As shown in Figure 4, the dynamics associated with collections of these "highly concentrated" vorticity fields often hold the key to understanding the underlying physics in the wake region associated with a wide range of flows. The mathematical representation of these concentrated vorticity regions as "particles", with dynamics driven by equation (24), has proven to be an exceptionally useful tool for understanding complex flows.

We now turn to two instructive examples in which the N-vortex system (24) can be solved explicitly.

Example 2. ($N = 2$) For the two-vortex problem, the equations (24) become:

$$\dot{z}_1^* = \frac{1}{2\pi i} \frac{\Gamma_2}{z_1 - z_2}, \tag{25}$$

$$\dot{z}_2^* = \frac{1}{2\pi i} \frac{\Gamma_1}{z_2 - z_1}. \tag{26}$$

Multiplying equation (25) by Γ_1 and equation (26) by Γ_2 and adding yields:

$$\Gamma_1 \dot{z}_1^* + \Gamma_2 \dot{z}_2^* = 0, \tag{27}$$

which tells us that $\Gamma_1 z_1 + \Gamma_2 z_2 = const$. This conserved quantity is known as the *center of vorticity* of the system. There is no loss of generality in choosing initial conditions $z_1(0)$ and $z_2(0)$ so that the center of vorticity is at the origin, hence we assume $\Gamma_1 z_1 + \Gamma_2 z_2 = 0$, or $z_2 = -(\Gamma 1/\Gamma 2)\, z_1$. If we substitute this into equation (25), we obtain:

$$z_1 \dot{z}_1^* = \frac{1}{2\pi i} \frac{\Gamma_2^2}{\Gamma_1 + \Gamma_2}. \tag{28}$$

Taking the complex conjugate of this equation yields:

$$z_1^* \dot{z}_1 = -\frac{1}{2\pi i} \frac{\Gamma_2^2}{\Gamma_1 + \Gamma_2}. \tag{29}$$

Adding these two equations tells us that $\| z_1 \|^2 = const.$, i.e., both vortices move on concentric circles around the center of vorticity of the two. More details can be found in [13].

Example 3. (Self-similar collapse) Self-similar solutions are those whose motion is constrained so that $z_\alpha(t) = \lambda_\alpha f(t)$, i.e. each vortex moves with the same function of time, $f(t)$. In equation (24), this yields:

$$\lambda_\alpha \dot{f} f^* = \frac{i}{2\pi} \sum \frac{\Gamma_\beta}{(\lambda_\alpha^* - \lambda_\beta^*)}. \tag{30}$$

Since the right-hand side is a (complex) constant, this is more easily written as:

$$\dot{f} f^* = A + iB. \tag{31}$$

Let $f(t) = r(t)\exp(i\theta(t))$, and separate real and imaginary parts to obtain:

$$\frac{1}{2}\frac{d(r^2)}{dt} = A; \quad r(0) = 1, \tag{32}$$

$$\frac{d\theta}{dt} = \frac{B}{r^2}. \tag{33}$$

These equations are easily solved to yield:

$$r(t) = \sqrt{2At + 1}, \tag{34}$$

$$\theta(t) = \frac{B}{2A}\log(2At + 1) + \theta(0) \quad (A \neq 0) \tag{35}$$

$$= Bt + \theta(0) \quad (A = 0). \tag{36}$$

If $A < 0$ (which can be chosen with appropriate initial conditions), then $r(t) \to 0$ in finite time, say $t = \hat{t}$, where $\hat{t} = -1/2A$. These are special solutions that spiral in logarithmically to a collision point in finite-time. See Figure 2.1 in [13] for a simulation of this self-similar spiral for $N = 3$.

3. CLOSING THOUGHTS

Complex variable theory drove progress in fluid mechanics in the 19th century, but now progress is driven more by advances in analysis techniques for nonlinear partial differential equations associated with the Navier–Stokes and Euler equations, dynamical systems techniques that focus on Lagrangian and transport aspects of fluid flow, and computational models that use distributed particle methods to represent vorticity fields. However, if put in the

proper context, complex variable theory still plays an important role in solving current scientific problems and lies at the heart of some cutting edge numerical techniques relying on boundary-integral formulations. The challenge is how to convey this to different audiences learning the subject. Striking the right balance between simplicity and complexity is at the heart of progress in science and mathematics, but the appropriate balance can be viewed differently by different audiences. To an audience of mathematicians taking their first course in complex variable theory, detailed discussions of the challenges of fluid mechanics would likely be a distraction and would fall on deaf ears. For these reasons, instructors typically quickly introduce Laplace's equation and potential flow, and run through the standard 19th-century list of fluid flow examples, all found in many introductory books on fluid mechanics, culminating in an example such as Example 1 described in this article. To an audience of physicists, engineers, and applied mathematicians taking their first course in fluid mechanics, regaling them with tales of the beauty and elegance of complex variable theory and its uses in fluid mechanics likely will lead to impatience, an impression that the examples are contrived and tortured, and accusations of striping all that is interesting out of a subject as rich, complex, and physically relevant as fluid mechanics. For these reasons, instructors often quickly introduce the basics of complex variables in cookbook fashion to cut to the chase. Off-loading some of the developmental aspects of these two subjects to Calculus II could be a partial solution to this problem, although structurally altering a Calculus syllabus at most large universities is easier said than done. Introducing simple and sequential excercises, such as the ones outlined in this article, in one or two lectures in a complex variable class, with a guest lecture given by an appropriate expert in fluid mechanics (most large research universities have several of these faculty members) may also be an honest, if not perfect way to convey the elegance of complex variable theory and its influence, both historical and current, on the subject of fluid mechanics.

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BIOGRAPHICAL SKETCH

Paul K. Newton received his B.S. in Applied Math/Physics at Harvard University and Ph.D. in Applied Mathematics from Brown University. After a post-doctoral fellowship at Stanford University, he was Assistant and Associate Professor of Mathematics and The Center for Complex Systems Research at the University of Illinois Champaign-Urbana. He has held visiting appointments at Caltech, Brown, Hokkaido University, The Kavli Institute for Theoretical Physics at U.C. Santa Barbara, and The Scripps Research Institute. He is currently Professor of Applied Math and Aerospace and Mechanical Engineering in the Viterbi School of Engineering at the University of Southern California. He serves as Editor-in-Chief of The Journal of Nonlinear Science, Advisor on Texts in Applied Mathematics Series, Springer-Verlag, New York. He has worked extensively on fluid mechanics and specializes in vortex dynamics and the Euler equations for fluid flow. He is the author of "The N-Vortex Problem: Analytical Techniques" published by Springer-Verlag.