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Point vortex dynamics in the post-Aref era

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Abstract

Three vortices move in a plane
You can solve that with limited pain
But add in a speck
And all goes to heck
Its motion is really insane!
(H Aref)

Hassan Aref's many contributions to the field of vorticity dynamics were highlighted in a dedicated lecture at the IUTAM Symposium on Vortex Dynamics: Formation, Structure and Function, Fukuoka Japan, 10–14 March 2013. Here we present a background discussion of the pleasures and challenges associated with working with discrete vorticity representations of fluid flows, one of the areas to which Professor Aref devoted his scientific career. This will lead to a discussion of several interesting problems that remain to be analyzed for the N-point vortex equations, including (i) dynamics and equilibria on curved surfaces; (ii) energy minimizing configurations (large N, general Γ_α); (iii) equilibria with defects; and (iv) formation of equilibria (in the presence of noise and other background flow). We finish with some final thoughts on the impact of Professor Aref's style and contributions to the field of low-dimensional discrete vortex modeling of fluid flows.

1. Introduction

The Navier–Stokes equations of incompressible fluid mechanics (Majda and Bertozzi 2002, Saffman 1992), an area of continuous fascination and focus of Hassan Aref throughout his

Figure 1. Hassan Aref (1950–2011).

scientific career from 1979–2011, are given by

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \Delta \vec{u}, \tag{1}$$

$$\nabla \cdot \vec{u} = 0, \tag{2}$$

$$\vec{u}(\vec{x}, 0) = \vec{f}(\vec{x}), \tag{3}$$

where $\vec{u}$ is the fluid velocity field, $-\nabla p$ is the pressure gradient, and ν is the kinematic viscosity (i.e. $\nu = \mu/\rho$, where μ is the dynamic viscosity and ρ is the fluid density). The two main terms of (1) which make the subject of incompressible fluid mechanics interesting are (i) the nonlinear inertial term $\vec{u} \cdot \nabla \vec{u}$, and (ii) the pressure gradient term $-\nabla p$, acting simultaneously in a dynamical setting, with initial conditions for the velocity field (3), and additional boundary conditions if one is modeling fluid–solid interactions.

To partially appreciate why these equations are challenging, decouple the pressure gradient term in (1) from the velocity field by taking the divergence of (1):

$$\left(\nabla \cdot \vec{u} \right)_t + \nabla \cdot \left(\vec{u} \cdot \nabla \vec{u} \right) = -\Delta p + \nu \Delta \left(\nabla \cdot \vec{u} \right), \tag{4}$$

then use incompressibility (2) to eliminate the first and last term, which gives rise to a Poisson equation for pressure:

$$\Delta p = -\nabla \cdot \left(\vec{u} \cdot \nabla \vec{u} \right). \tag{5}$$

From this, we can invert the Laplacian using the appropriate Green's function G, and write the pressure explicitly in terms of the velocity field:

$$p = -\int G \nabla \cdot \left(\vec{u} \cdot \nabla \vec{u} \right) \mathrm{d}V. \tag{6}$$

Plugging (6) back into (1) gives the decoupled velocity field equation as an alternative to (1):

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = \nabla \left(\int G \nabla \cdot \left(\vec{u} \cdot \nabla \vec{u} \right) \mathrm{d}V \right) + \nu \Delta \vec{u}, \tag{7}$$

showing clearly the nonlocal nature of the pressure gradient term. Once this equation is solved for a given initial velocity field $\vec{u}(\vec{x}, 0)$, the result can be used in (6) to obtain the pressure. Thus, the pressure is fully determined once the velocity field is known, but because of (6), one must know the *global* velocity field in order to determine it. The *global* nature of the equations and the *nonlinear coupling* between the pressure gradient and the velocity field, not so immediately evident in the formulation (1)–(3), represents a significant challenge to

any attempts at modelling and simulating all but the most well-conceived flow geometries and initial conditions. This is one of the motivating reasons why low-dimensional models of more complex flows of interest present continued added value to our understanding of incompressible fluid mechanics, approximate as they often are.

A useful re-formulation, often invoked, is to introduce the vorticity field induced by the velocity field, $\vec{\omega} = \nabla \times \vec{u}$, whose equation is obtained by 'curling' (1):

$$\vec{\omega}_t + \vec{u} \cdot \nabla\vec{\omega} = \vec{\omega} \cdot \nabla\vec{u} + \nu\Delta\vec{\omega}, \tag{8}$$

which formally eliminates the pressure term from (1), but there is no free lunch. The price one pays for this is that (8) is now coupled to the Biot–Savart formula, analogous to the relationship between the magnetic field and the electric current which generates it (Jackson 1962):

$$\vec{u} = -\frac{1}{4\pi} \int \frac{(\vec{x} - \vec{z}) \times \vec{\omega}(\vec{z})\,\mathrm{d}\vec{z}}{\parallel \vec{x} - \vec{z} \parallel^3}, \tag{9}$$

which relates the vorticity distribution to the induced velocity field. The solution to these equations are then used in the right side of (6) to obtain p.

The 'vortex-stretching' term (Majda and Bertozzi 2002, Saffman 1992) on the right-hand side of (8), $\vec{\omega} \cdot \nabla\vec{u}$, only acts in three dimensions, since in two-dimensional problems, the curl of the velocity field is orthogonal to the gradient of the velocity field. In the inviscid 'Euler' limit $\nu \to 0$, one is left with the pure vorticity dynamics equations, which in R^2 become

$$\omega_t + \vec{u} \cdot \nabla\omega = 0, \quad \left(\omega(\vec{x}, t) \in R^1\right), \tag{10}$$

and in R^3 become

$$\vec{\omega}_t + \vec{u} \cdot \nabla\vec{\omega} = \vec{\omega} \cdot \nabla\vec{u}, \quad \left(\vec{\omega}(\vec{x}, t) \in R^3\right). \tag{11}$$

Based on the structure of equation (10), one often says that in two dimensions, vorticity 'goes with the flow', i.e. is transported in a Lagrangian way like a material element (Newton 2001, Saffman 1992). In fact, there is more. It can also be proven quite easily that in two dimensions, *any* smooth function of the vorticity, $f(\omega)$, is conserved under the flow (Newton 2001, Saffman 1992). Thus, vorticity not only goes with the flow, but is a conserved quantity carried by the flow.

A particularly simple and useful way to 'carry' the conserved vorticity field is obtained by concentrating the scalar vorticity field $\omega(\vec{x}, t)$ in planar regions that are highly localized:

$$\omega(\vec{x}) = \sum_{i=1}^{N} \frac{\Gamma_i}{2\pi} \phi_\epsilon(\vec{x} - \vec{x}_i), \tag{12}$$

$$\phi_\epsilon(\vec{x}) = \frac{1}{\epsilon^2}\phi\left(\frac{\vec{x}}{\epsilon}\right), \quad \epsilon \ll 1, \tag{13}$$

where ϕ is any suitably normalized radially symmetric function with the property $\int \phi \mathrm{d}\vec{x} = 1$. One popular and useful representation is the point vortex approximation $\phi_\epsilon(\vec{x}) = \delta(\vec{x})$, where δ is the Dirac delta function. This turns the field equation (10) into a Hamiltonian particle interaction problem (Newton 2001) which is analytically attractive, although is not typically the most optimal way to proceed numerically (Cottet and Koumoutsakos 2000) because of the divergent velocity field approaching each point vortex. It is not difficult to show that such a singularity located at the origin $\vec{x} = 0$, gives rise to a planar velocity field:

$$u_r = 0, \tag{14}$$

$$u_\theta = \frac{\Gamma}{2\pi r}, \tag{15}$$

where $\frac{\Gamma}{2\pi}$ represents the 'circulation' that the velocity field carries. Written in complex variable form, the velocity field for such a point vortex located at $z = 0$ is given by

$$\dot{z}^* = \frac{\Gamma}{2\pi i}\frac{1}{z}, \quad z(t) = x(t) + iy(t). \tag{16}$$

Here, z^* denotes the complex conjugate of z. A collection of N of these point vortices, each located at $z = z_\beta(t)$, with strengths Γ_β, $\beta = 1, ..., N$, by linear superposition, produces the field

$$\dot{z}^* = \frac{1}{2\pi i}\sum_{\beta=1}^{N}\frac{\Gamma_\beta}{z - z_\beta}. \tag{17}$$

Then, if we advect each of the point vortices by the velocity field generated by all of the others, as (10) tells us to do, we arrive at the N-vortex dynamical system:

$$\dot{z}_\alpha^* = \frac{1}{2\pi i}\sum_{\beta=1}^{N}{}'\frac{\Gamma_\beta}{z_\alpha - z_\beta}, \tag{18}$$

where $'$ indicates that $\beta \neq \alpha$. These are the famous N-point vortex equations of motion (Newton 2001). Each of the vortex strengths, Γ_β, are constants, as dictated by the conservation of vorticity, and they each are advected by the velocity field generated by all of the others, as dictated by (10). These seemingly simple equations were a continual and renewed source of inspiration for Hassan Aref from the time of his PhD dissertation 'Turbulence and vortex dynamics in two dimensions' (Cornell University 1980, PhD Supervisor E Siggia), returning to them throughout his career. The limerick quoted at the front of this article (which I found in my collection of notes, the 'Zakopane Limericks', while preparing this review) was written by Hassan in reference to his famous 1979 and 1982 papers (Aref 1979, Aref and Pomphrey 1982) which remain classics in the field.

What makes this 'Lagrangian' idea of discrete vortices so attractive is that it allows a particle viewpoint to be adopted for a flow field that is otherwise viewed in terms of pressure gradients, accelerations, or as a collection of interacting waves. One can think of the flow field as a 'gas' of interacting particles (as was implicit in the pioneering statistical approach of Onsager (Eyink and Sreenivasan 2006), albeit particles that sometimes have considerable intrinsic structure. This 'has the virtue,' as Charney so eloquently put it, 'that mass, energy, linear and angular momentum continue to be conserved, and that the motions represented are those of conceivable, though idealized, physical systems' (Charney 1963). One works with spatially localized lumps of rotational fluid moving in a background flow that is, by comparison, essentially irrotational. The representation of a flow as a collection of discrete vortices, particularly a finite collection of such vortices, will not always be reasonable but when it is, it provides an economical, physically motivated, and analytically interesting description of the flow. When it applies, it opens up connections to bodies of physical theory and mathematics not easily brought into the analysis of fluid flows otherwise (Aref 2007b). The somewhat elusive connection between solutions of the Euler equations ($\nu = 0$) and the large Reynolds number limit ($Re \sim \frac{1}{\nu} \to \infty$) of the Navier–Stokes equations lies at the heart of the turbulence problem (Eyink and Sreenivasan 2006, Saffman 1992). Since long-lived

concentrated vortex structures appear in solutions of the Navier–Stokes equations, and since persistent, concentrated vortices can be embedded in an inviscid flowfield, it is a natural assumption that tracking such concentrated vortices in the Euler equations can reveal properties of solutions to the Navier–Stokes equations in the high Reynolds number limit and it is primarily this simple observation for the sustained interest in vortex dynamics over the past 150 years, since the seminal paper by von Helmholtz in 1858. A comprehensive bibliography of the field of vortex dynamics from 1858–1956 was compiled by Meleshko and Aref (2007). Much of Hassan Aref's career was devoted to fleshing out and developing the details and subtleties associated with these connections. A secondary goal of this article is to compile a reasonably complete bibliography of Hassan Aref's lifetime publications. While we cannot claim the reference section is exhaustively complete, it does contain his most influential publications in the field of vortex dynamics, and should be an excellent starting point for any student interested in learning the foundations of the field.

2. Some interesting problems

Hassan Aref's work in the field of vortex dynamics touched many areas, some of which need further development with low-hanging fruit still to be picked. Here is a sampling of interesting topics.

1. **Dynamics and equilibria on curved surfaces.** An interesting recent discussion on the general problem of vortex configurations and dynamics on two-dimensional curved surfaces can be found in Turner, Vitelli, and Nelson's article (Turner *et al* 2010). Their focus is on how the imposed geometric deformations, such as a curved surface, couples to the in-plane order as evidenced by the changes in allowable crystal vortex lattice patterns. They emphasize work done in the context of thin layers of superfluid liquid helium, which of course adds the additional quantization constraint on the vortex circulations not always imposed in other applications. By far the largest literature available is on constant curvature surfaces, such as the sphere (Barreiro *et al* 2010, Chamoun *et al* 2009, Crowdy 2004, Jamaloodeen and Newton 2006, Kidambi and Newton 1998, Laurent-Polz 2002, Lim *et al* 2001, Newton 2001, Newton and Ross 2006, Newton and Sakajo 2011, 2009, 2007, Newton and Shokraneh 2008, 2006, Newton and Ostrovskyi 2012, O'Neil 2008, 2012) where direct applications to large-scale atmospheric and oceanographic flows are apparent (Newton 2010). There is also new and interesting work relating equilibrium point vortex configurations on the sphere with limiting solutions of related partial differential equations such as the Gross–Pitaevskii equations (Gelantalis and Sternberg 2012). There is other earlier work that considers the dynamics of point vortices on more general surfaces of symmetry, such as Hally's paper (Hally 1980) on surfaces of revolution, and Kimura's work (Kimura 1999) on surfaces of constant curvature, as well as others (Boatto and Koiller 2008, Hwang and Kim 2013, Montaldi *et al* 2003, San Miguel 2013, Soulière and Tokieda 2002). In the plane, one of the most fruitful approaches has been to connect the distribution of point vortices along a straight line, in equilibrium, to the zeroes of classes of special functions (Aref *et al* 2003, Aref 2007c), a topic that Hassan Aref developed extensively. On curved surfaces, the analogous question is to look for equilibria that lie along geodesic curves, such as great circles on the sphere (Aref *et al* 2003), and ask for the distribution of point vortex positions. This topic is essentially unexplored, although see the papers of Kidambi and Newton (1998) and O'Neil (2008), particulary in the case where the vortex strengths are

not all equal—even for collinear equilibria in the plane. A separate topic is to identify the analogue of Abrikosov lattices (Abrikosov 2004) on curved surfaces, or in general, special configurations of equilibria with discrete symmetries, like the Platonic solid family on the sphere (Jamaloodeen and Newton 2006, Newton and Ostrovskyi 2012, Chamoun *et al* 2009). Aside from the physics associated with these configurations with discrete symmetries that reflect the geometry of the surface and the vortex circulations, they have played a role in the identification of optimal grid generation on surfaces in the computer graphics and numerical analysis communities where particle interaction methods are used to distribute grid points optimally (minimal energy configurations) across a curved surface (Zhong *et al* 2013).

2. **Energy minimizers for large *N*.** The general research program of identifying the lowest energy configurations for each value of *N* was started ambitiously and explicitly by Campbell and Ziff (1978, 1979), papers that were among Hassan Aref's favorites, from which he drew continual and renewed inspiration. An interesting lesser known paper along these lines is that of Gueron and Shafrir (1999). Simply stated, the problem here is given a value of *N*, and given a collection of values of vortex circulations $\left(\Gamma_1, \Gamma_2,...,\Gamma_N \right)$ (usually taken to be all equal), find the stationary or relative equilibrium positions of the point vortices governed by (18) that minimizes the Hamiltonian energy of the configuration,

$$H = -\frac{1}{4\pi} \sum_{\beta=1}^{N} \sum_{\alpha=1}^{N} {}' \Gamma_\alpha \Gamma_\beta \log \left| z_\alpha - z_\beta \right|, \tag{19}$$

subject to the constraints of the remaining conserved quantities (discussed in Aref *et al* 2003). The works of Campbell and Ziff (1978, 1979) take a direct computational approach to identifying the list of two-dimensional patterns that correspond to minima in the free energy of a system of identical point vortices in the plane, for values of $N = 1, 2,..., 30, 37, 50$. Identifying these configurations as energy minimizers kills two birds with one stone. One obtains not only the configuration, but also its stability properties, which otherwise would have to be obtained by separate methods (Aref 1995, 2009). There is much more one can do along these lines, particularly for larger *N* values (say $N > 100$), and general vortex circulations Γ. The larger the value of *N*, the more attractive the linear algebra formulation of Newton and Chamoun (2009) becomes. The key to this approach is to directly formulate the *N*-vortex equations in terms of their inter-particle distances $l_{\alpha\lambda}^2 = \| \vec{x}_\alpha - \vec{x}_\lambda \|^2$ without yet committing to the values of the circulations $\left(\Gamma_1, \Gamma_2,...,\Gamma_N \right)$, leaving them, for the moment, as free parameters. The inter-vortical distances remain constant for configurations that are locked in either fixed or relative equilibrium patterns. The dynamical equations for the $N(N-1)/2$ equations for $l_{\alpha\lambda}$ become

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(l_{\alpha\lambda}^2 \right) = \frac{2}{\pi} \sum_{\beta=1}^{N} {}'' \Gamma_\beta A_{\alpha\lambda\beta} \left(\frac{1}{l_{\alpha\beta}^2} - \frac{1}{l_{\lambda\beta}^2} \right), \quad (\alpha, \lambda = 1, ..., N). \tag{20}$$

The notation $''$ indicates that the sum is over all values of β, except for $\beta = \alpha$ and $\beta = \lambda$, and $A_{\alpha\lambda\beta} = \frac{1}{2} \| \vec{x}_\alpha \times \vec{x}_\beta + \vec{x}_\lambda \times \vec{x}_\alpha + \vec{x}_\beta \times \vec{x}_\lambda \|$ is the area subtended by $\left(\vec{x}_\alpha, \vec{x}_\lambda, \vec{x}_\beta \right)$. Since all fixed and relative equilibria have $l_{\alpha\lambda}^2 = $ const, equilibria are *fixed points* of equation (20), which can be written as a rectangular matrix system:

$$A\vec{\Gamma} = 0, \tag{21}$$

where the configuration matrix $A \in R^{M \times N}$ has N columns and $M = N(N-1)/2$ rows, with $\vec{\Gamma} = (\Gamma_1, \Gamma_2, ..., \Gamma_N)$. The configuration matrix encodes the *geometry* of the pattern, and only those patterns that give rise to configuration matrices with nontrivial nullspaces correspond to equilibria. Thus, existence of an equilibrium solution hinges on whether or not $det(A^T A) = 0$, while uniqueness of the choice of values for $\vec{\Gamma}$ (up to multiplicative constants) depends on Rank(A). Once the nullspace vector $\vec{\Gamma}$ is obtained for a given configuration, it can be used *a posteriori* to calculate the Hamiltonian energy (19). It is the combined properties of the configuration matrix A and the Hamiltonian energy H that determine all of the salient properties of the equilibrium in question, such as uniqueness, energy levels and stability. The larger the value of N, the less tractable the *explicit*, classical formulations become, and the more appealing this linear algebra approach becomes, since there are a host of techniques available in this field, like the singular value decomposition, that can be used. See Newton and Chamoun (2009) and Barreiro *et al* (2010) for more details and examples.

Characterizing the energy levels, as was done in Campbell and Ziff's papers (Campbell and Ziff 1978, 1979), and in Hassan Aref's recent work (Aref 2007a), for minimum energy configurations (for each N) remains a challenge. When N is large (the definition of what constitutes *large* is a moving target, but let us take as a working definition values of N larger than those considered in Campbell and Ziff (1978, 1979), there are many equilibria for each value of N, the spacing of their energies gets tighter and tighter, the differences in their geometry become increasingly subtle to detect, hence alternative characterizations, such as their *entropy* based on the spectrum of the configuration matrix A, become a useful tool (Newton and Chamoun 2009).

3. **Equilibria with defects.** Point vortex equilibria with defect structures have been identified beautifully in experiments involving Bose–Einstein condensates (see figure 1.6 from Newton and Chamoun (2009), but these equilibria, which generally arise because they sit at lower energy levels than their symmetric counterparts nearby (see the relevant paper, Campbell (1990)) have yet to be derived as explicit solutions to the N-vortex equations (18). Aref's nice paper (Aref and Vainchtein 1998) identifies a whole host of asymmetric equilibria, an observation that at the time was somewhat surprising to many in the field. How exactly to systematically categorize the asymmetric equilibria or identify those that can be interpreted as equilibria with defects is not yet clear, but again, the linear algebra formulation may be very useful for these purposes. Imagine, for example, a symmetric, defect-free configuration with configuration matrix A_0, and strength vector $\vec{\Gamma}_0$. Look for a nearby configuration A_ϵ, with strength vector $\vec{\Gamma}_\epsilon$. Hence, let

$$A_\epsilon \equiv A_0 + A_1, \quad \text{where} \quad A_1 \equiv (A_\epsilon - A_0), \tag{22}$$

$$\vec{\Gamma}_\epsilon \equiv \vec{\Gamma}_0 + \vec{\Gamma}_1, \quad \text{where} \quad \vec{\Gamma}_1 \equiv (\Gamma_\epsilon - \vec{\Gamma}_0), \tag{23}$$

and assume that the defect structure is a 'perturbation' of a defect-free structure:

$$\frac{\| A_1 \|}{\| A_0 \|} \sim \epsilon; \quad \frac{\| \vec{\Gamma}_1 \|}{\| \vec{\Gamma}_0 \|} \sim \epsilon. \tag{24}$$

The goal, then, is to find equilibria with defects corresponding to the equation

$$A_{\epsilon}\vec{\Gamma}_{\epsilon} = \left(A_0 + A_1\right)\left(\vec{\Gamma}_0 + \vec{\Gamma}_1\right) = 0. \qquad (25)$$

This sets up the hierarchy of equations

$$A_0\vec{\Gamma}_0 = 0, \qquad (26)$$

$$A_0\vec{\Gamma}_1 = -A_1\vec{\Gamma}_0, \qquad (27)$$

$$A_1\vec{\Gamma}_1 = 0. \qquad (28)$$

Equation (26) is automatically satisfied if the unperturbed 'defect-free' state is an equilibrium. Equation (27), by the Fredholm alternative, requires that

$$-A_1\vec{\Gamma}_0 \perp Null\left(A_0^T\right), \qquad (29)$$

which puts a constraint on allowable A_1 matrices, given A_0 and $\vec{\Gamma}_0$. Equation (28) then pins down the vector $\vec{\Gamma}_1$:

$$\vec{\Gamma}_1 \in Null\left(A_1\right). \qquad (30)$$

An additional requirement would be that the perturbed energy (19) is lower than the unperturbed energy, i.e. $H_{\epsilon} < H_0$. Using these ideas to identify structures with defects of the type found in nature (point defects, grain-boundary defects) would be new and quite interesting.

4. **Formation of equilibria (in the presence of noise).** In reality, equilibria form from initial conditions that do not correspond exactly to an equilibrium state, with 'background' dynamics that introduces stochasticity to an already chaotic deterministic problem (Aref 1983). The *formation dynamics* of equilibria in the presence of noise is an outstanding issue discussed nicely in Fine *et al* (1995), and Durkin *et al* (2000)—see figure 1.14 from Newton and Chamoun (2009) for an example of one of these studies. The basic equations were first described in Osada's 1985 paper (Osada 1985), while a more thorough approach for analyzing stochastic versions of the N-vortex equations were developed more recently in Kotelenez (1995) and in the PhD thesis of Rath (2010). General techniques associated with stochastic vorticity fields are discussed thoroughly in Majda and Wang's book (Majda and Wang 2006) in the context of geophysical flows. But to our knowledge, no one has begun looking at the process of formation of fixed and relative equilbria in the presence of a stochastic field, a topic potentially rich with possibilities and physical relevance.

There are, of course, many other interesting topics currently under development by many people in the vortex dynamics community, some of whom participated in the wonderful IUTAM Symposium in Fukuoka Japan, 10–14 March 2013. The topics discussed in this review are the ones this author feels would be closest to the interests of Hassan Aref, hence of special relevance to the workshop in his honor.

3. Final thoughts

The passion, single-minded devotion, high standards and taste that Hassan Aref brought to the field of vortex dynamics, and mechanics in general, have led some to describe him as 'a 19th century physicist who happened to live into the 21st century'[1]. His papers combine physical

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[1] Quote taken from Prof. Alan Needleman's memories on the Aref Memorial website: hassan.mikearef.com

insights, analytical elegance, didactic precision and clarity, and a sense of mastery of the subject that are unique. The references cited below to his works offer the reader a glimpse, but are by no means a complete bibliography of all of his many contributions to this field. Many of the techniques developed over the course of his career have layed the groundwork for the use of discrete low-dimensional vortex models whose nonlinear dynamics can be understood far more completely than alternative techniques offer and we expect this now established sub-field of fluid mechanics to continue to contribute new insights and to provide a practical starting point for the development of more complex and targeted analytical models of complex fluid flows.

References

Abrikosov A A 2004 Type II superconductors and the vortex lattice *Rev. Mod. Phys.* **76** 975–9

Aref H 1979 Motion of three vortices *Phys. Fluids* **22** 393

Aref H 1982 Point vortex motions with a center of symmetry *Phys. Fluids* **25** 2183

Aref H 1983 Integrable, chaotic, and turbulent vortex motion in two-dimensional flows *Ann. Rev. Fluid Mech.* **15** 345–89

Aref H 1984 Stirring by chaotic advection *J. Fluid Mech.* **143** 1–21

Aref H 1986 The numerical experiment in fluid mechanics *J. Fluid Mech.* **173** 15–41

Aref H 1990 Chaotic advection of fluid particles *Phil. Trans. R. Soc.* A **333** 273–88

Aref H 1991 Stochastic particle motion in laminar flows *Phys. Fluids* A **3** 1009

Aref H 1995 On the equilibrium and stability of a row of point vortices *J. Fluid Mech.* **290** 167–82

Aref H 1999 Fluid dynamics: order in chaos *Nature* **401** 756–8

Aref H 2002a The development of chaotic advection *Phys. Fluids* **14** 1315

Aref H 2002b A transformation of the point vortex equations *Phys. Fluids* **14** 2395

Aref H 2007a Point vortex dynamics: a classical mathematics playground *J. Math. Phys.* **48** 065401

Aref H 2007b A note on the energy of relative equilibria of point vortices *Phys. Fluids* **19** 103603

Aref H 2007c Vortices and polynomials *Fluid Dyn. Res.* **39** 5–23

Aref H 2009 Stability of relative equilibria of three vortices *Phys. Fluids* **21** 094101

Aref H 2010 Self-similar motion of three point vortices *Phys. Fluids* **22** 057104

Aref H 2011 Relative equilibria of point vortices and the fundamental theorem of algebra *Phil. Trans. R. Soc.* A **467** 2168–84

Aref H and Brons M 1998 On stagnation points and streamline topology in vortex flows *J. Fluid Mech.* **370** 1–27

Aref H and Flinchem E P 1984 Dynamics of a vortex filament in a shear flow *J. Fluid Mech.* **148** 477–97

Aref H and Jones S W 1993 Chaotic motion of a solid through ideal fluid *Phys. Fluids* A **5** 3026

Aref H and Jones S W 1999 Enhanced separation of diffusing particles by chaotic advection *Phys. Fluids* A **1** 470

Aref H, Kadtke J B, Zawadzki I, Campbell L J and Eckhardt B 1988 Point vortex dynamics: recent results and open problems *Fluid Dyn. Res.* **3** 63–74

Aref H, Newton P K, Stremler M A, Tokieda T and Vainchtein D L 2003 Vortex crystals *Adv. Appl. Mech.* **39** 1–79

Aref H and Pomphrey N 1980 Integrable and chaotic motions of four vortices *Phys. Lett.* A **78** 297–300

Aref H and Pomphrey N 1982 Integrable and chaotic motions of four vortices. I. The case of identical vortices *Proc. R. Soc. London* A **380** 359–87

Aref H, Rott N and Thomann H 1992 Gröbli's solution of the three-vortex problem *Ann. Rev. Fluid Mech.* **24** 1–20

Aref H and Siggia E D 1980 Vortex dynamics of the two-dimensional turbulent shear layer *J. Fluid Mech.* **100** 705–37

Aref H and Siggia E D 1981 Evolution and breakdown of a vortex street in two dimensions *J. Fluid Mech.* **109** 435–63

Aref H and Stremler M A 1996 On the motion of three point vortices in a periodic strip *J. Fluid Mech.* **314** 1–26

Aref H and Stremler M A 1999 Four-vortex motion with zero total circulation and impulse *Phys. Fluids* **11** 3704

Aref H and Vainchtein D L 1998 Point vortices exhibit asymmetric equilibria *Nature* **392** 769–70

Aref H and van Buren M 2005 Vortex triple rings *Phys. Fluids* **17** 057104

Aref H and Zawadzki I 1991 Linking of vortex rings *Nature* **354** 50–53

Barreiro A, Bronski J and Newton P K 2010 Spectral gradient flow and equilibrium configurations of point vortices *Proc. R. Soc. London* A **466** 1687–702

Beebe D J, Adrian R J, Olsen M G, Stremler M A, Aref H and Jo B H 2001 Passive mixing in microchannels *Fabr. Flow Exp. Mecanique Ind.* **2** 343–8

Boatto S and Koiller J 2008 Vortices on closed surfaces　arXiv:0802.4313v1

Boyland P, Aref H and Stremler M A 2000 Topological fluid mechanics of stirring *J. Fluid Mech.* **403** 277–304

Boyland P, Stremler M and Aref H 2003 Topological fluid mechanics of point vortex motions *Physica* D **175** 69–95

Campbell L J 1990 Displaced vortices on lattices *J. Phys. A: Math. Gen.* **23** 3317

Campbell L J and Ziff R 1978 A catelogue of 2D vortex patterns *Los Alamos Sci. Rep.* LA-7384-MS

Campbell L J and Ziff R 1979 Vortex patterns and energies in a rotating superfluid *Phys. Rev.* B **20** 1886–902

Chamoun G, Kanso E and Newton P K 2009 Von Kàrmàn vortex streets on a sphere *Phys. Fluids* **21** 116603

Charney J 1963 Numerical experiments in atmospheric hydrodynamics in experimental arithmetic, high speed computing and mathematics *Proc. Symp. Applied Mathematics* vol 15 (Providence, RI: American Mathematical Society) p 289310

Clarkson P A 2009 Vortices and polynomials *Stud. Appl. Math.* **123** 37–62

Cottet G H and Koumoutsakos P D 2000 *Vortex Methods: Theory and Practice* (Cambridge: Cambridge University Press)

Crowdy D 2004 Stuart vortices on a sphere *J. Fluid Mech.* **498** 381–402

Dirksen T and Aref H 2011 Close pairs of relative equilibria for identical point vortices *Phys. Fluids* **23** 051706

Durkin D and Fajans J 2000 Experiments on two-dimensional vortex patterns *Phys. Fluids* **12** 289–93

Eyink G L and Sreenivasan K R 2006 Onsager and the theory of hydrodynamic turbulence *Rev. Mod. Phys.* **78** 87–135

Fine K S, Cass A C, Flynn W G and Driscoll C F 1995 Relaxation of 2D turbulence to vortex crystals *Phys. Rev. Lett.* **75** 3277–80

Gelantalis M and Sternberg P 2012 Rotating 2N-vortex solutions to the Gross–Pitaevskii equation on S^2 *J. Math. Phys.* **53** 083701

Gueron S and Shafrir I 1999 On a discrete variational problem involving interacting particles *SIAM J. Appl. Math.* **60** 1–17

Hally D 1980 Stability of streets of vortices on surfaces of revolution with a reflection symmetry *J. Math. Phys.* **21** 211–7

Herdtle T and Aref H 1992 Numerical experiments on two-dimensional foam *J. Fluid Mech.* **241** 233–60

Hwang S and Kim S K 2013 Relative equilibria of point vortices on the hyperbolic sphere *J. Math. Phys.* **54** 063101

Jackson J D 1962 *Classical Electrodynamics* (New York: Wiley)

Jamaloodeen M I and Newton P K 2006 The *N*-vortex problem on a rotating sphere II. Heterogeneous platonic solid equilibria *Proc. R. Soc. London* A **462** 3277–99

Jones S W and Aref H 1988 Chaotic advection in pulsed sourcesink systems *Phys. Fluids* **31** 469

Jones S W, Thomas O M and Aref H 1989 Chaotic advection by laminar flow in a twisted pipe *J. Fluid Mech.* **209** 335–57

Kidambi R and Newton P K 1998 Motion of three point vortices on a sphere *Physica* D **116** 143–75

Kimura Y 1999 Vortex motion on surfaces with constant curvature *Proc. R. Soc. London* A **455** 245–59

Kotelenez P 1995 A stochastic Navier–Stokes equation for the vorticity of a two-dimensional fluid *Ann. App. Prob.* **5** 1126–60

Laurent-Polz F 2002 Point vortices on the sphere: a case with opposite vorticities *Nonlinearity* **15** 143

Lim C, Montaldi J and Roberts M 2001 Relative equilibria of point vortices on the sphere *Physica* D **148** 97–135

Liu R H, Stremler M A, Sharp K V, Olsen M G, Santiago J G, Adrian R J, Aref H and Beebe D J 2000 Passive mixing in a three-dimensional serpentine microchannel *J. Micro Syst.* **9** 190–19

Majda A J and Bertozzi A L 2002 *Vorticity and Incompressible Flow (Cambridge Texts in Applied Mathematics)* (Cambridge: Cambridge University Press)

Majda A J and Wang X 2006 *Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows (Cambridge Texts in Applied Mathematics)* (Cambridge: Cambridge University Press)

Meleshko V V and Aref H 2007 A bibliography of vortex dynamics 1858–1956 *Adv. Appl. Mech.* **41** 197–292

Montaldi J, Soulière A and Tokieda T 2003 Vortex dynamics on a cylinder *SIAM J. Appl. Dyn. Syst.* **2** 417–30

Newton P K 2001 *The N-Vortex Problem: Analytical Techniques* vol 101 (New York: Springer)

Newton P K 2010 The *N*-vortex problem on a sphere: geophysical mechanisms that break integrability *Theor. Comput. Fluid Mech.* **24** 137–49

Newton P K and Chamoun G 2009 Vortex lattice theory: a particle interaction perspective *SIAM Rev.* **51** 501–42

Newton P K and Ross S D 2006 Chaotic advection in the restricted four-vortex problem on a sphere *Physica* D **223** 36–53

Newton P K and Sakajo T 2007 The *N*-vortex problem on a rotating sphere III. Ring configurations coupled to a background field *Proc. R. Soc. London* A **463** 961–77

Newton P K and Sakajo T 2009 Point vortex equilibria on the sphere via Brownian ratchets *Proc. R. Soc. London* A **465** 437–55

Newton P K and Sakajo T 2011 Point vortex equilibria and optimal packings of circles on a sphere *Proc. R. Soc. London* A **467** 1468–90

Newton P K and Shokraneh H 2006 The *N*-vortex problem on a rotating sphere I. Multi-frequency configurations *Proc. R. Soc. London* A **462** 149–69

Newton P K and Shokraneh H 2008 Interacting dipole pairs on a rotating sphere *Proc. R. Soc. London* A **464** 1525–41

Newton P K and Ostrovskyi V 2012 Energy-momentum stability of icosahedral configurations of point vortices on a sphere *J. Non. Sci.* **22** 499–515

O'Neil K A 1987 Stationary configurations of point vortices *Trans. AMS* **302** 383–425

O'Neil K A 2006 Minimal polynomial systems for point vortex equilibria *Physica* D **219** 69–79

O'Neil K A 2008 Relative equilibria of point vortices that lie on a great circle of a sphere *Nonlinearity* **21** 2043

O'Neil K A 2012 Stationary states of identical point vortices and vortex foam on the sphere *Proc. R. Soc. London* A **469** 20120622

Osada H 1985 A stochastic differential equation arising from the vortex problem *Phys. Rev. Lett.* **93** 84501

Rath M 2010 Stochastic point vortex dynamics *PhD Thesis* Westfälische-Wilhelms-Universität Münster, Germany

Roenby J and Aref H 2010 Chaos in body–vortex interactions *Proc. R. Soc. London* A **466** 1871–91

Saffman P G 1992 *Vortex Dynamics* (Cambridge: Cambridge University Press)

San Miguel A 2013 Numerical description of the motion of a point vortex pair on ovaloids *J. Phys. A: Math. Theor.* **46** 115502

Siggia E D and Aref H 1981 Point vortex simulation of the inverse energy cascade in two-dimensional turbulence *Phys. Fluids* **24** 171

Stremler M A and Aref H 1999 Motion of three point vortices in a periodic parallelogram *J. Fluid Mech.* **392** 101–28

Stremler M A, Haselton F R and Aref H 2004 Designing for chaos: applications of chaotic advection at the microscale *Phil. Trans. R. Soc.* A **362** 1019–36

Snelders H A M 1976 A M Meyer's experiments with floating magnets and their use in the atomic theories of matter *Ann. Sci.* **33** 67–80

Soulière A and Tokieda T 2002 Periodic motions of vortices on surfaces with symmetry *J. Fluid Mech.* **460** 83–92

Turner A M, Vitelli V and Nelson D R 2010 Vortices on curved surfaces *Rev. Mod. Phys.* **82** 1301–48

von Helmholtz H 1858 On the integrals of the hydrodynamical equations which express vortex motion *Phil. Mag.* **4** 485–512

Yarmchuk E R, Gordon M J V and Packard R E 1979 Observation of stationary vortex arrays in rotating superfluid helium *Phys. Rev. Lett.* **43** 214–8

Zhong Z, Guo X, Wang W, Levy B, Sun F, Liu Y and Mao W 2013 Particle-based anisotropic surface meshing *Preprint* alice.loria.fr/publications/papers/2013/PASM/PASM.pdf