## INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand comer and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality $6^{\prime \prime} \times 9^{\prime \prime}$ black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

A Bell \& Howell Information Company<br>300 North Zeeb Road, Ann Arbor MI 48106-1346 USA<br>313/761-4700 800/521-0600

# THE INTERACTION OF SHOCK WAVES 

 AND DISPERSIVE WAVESBY<br>RALPH MARTIN AXEL<br>B.S., University of Massachusetts, 1990<br>M.S., University of Illinois, 1994

THESIS
Submitted in partial fullfillment of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate College of the
University of Ilinois at Urbana-Champaign, 1996

Urbana, Illinois

UMI Microform 9702452
Copyright 1996, by UMI Company. All rights reserved.
This microform edition is protected against unauthorized copying under Title 17, United States Code.

## UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

$\qquad$
THE GRADUATE COLLEGE

April, 1996

WE HEREBY RECOMMEND THAT THE THESIS BY
RALPH MARTIN AXEL


Committee on Final Examination $\dagger$

$\dagger$ Required for doctor's degree but not for master's.
$0.51=$

## Abstract

We introduce and analyze a coupled system of partial differential equations which model the interaction of shocks with a dispersive wave envelope. The system mimics the Zakharov equations from weak plasma turbulence theory but replaces the linear wave equation in that system by a nonlinear equation allowing the formation of shocks. This nonlinear equation is a hyperbolic conservation law forced by the dispersive wave. Chapter 1 considers a weak coupling in which the nonlinear wave evolves independently but appears as the potential in the time dependent Schrödinger equation governing the dispersive wave. We solve the Riemann problem by constructing solutions to the Schrödinger equation that are steady in a frame of reference moving with the shock. As in nonlinear WKB approaches, the Schrödinger equation is separated into phaseamplitude form. A viscous diffusion term is then added to the shock equation and by explicitly constructing asymptotic expansions in the small diffusion coefficient, we show that the Riemann problem steady states are zero diffusion limits of the regularized problem. The expansions are unusual in that it is necessary to keep track of exponentially small terms to obtain algebraically small terms. We then construct a family of time dependent solutions in the case that the initial data for the nonlinear wave equation evolves to a steady-state shock in finite time. We prove that the shock formation drives a finite time blow-up in the phase gradient of the dispersive wave, and identify a family of transient solutions of the Riemann problem steady states. In Chapter 2, the incompressible limit of the fully coupled equations is considered. In this limit, the ratio of the shock speed to the group velocity is large: as the coefficient of the time derivative of the shock wave goes to zero, the wavespeed becomes infinite. When this coefficient equals zero, the full system reduces to the single linear or nonlinear Schrödinger equation for the dispersive wave envelope. After presenting some exact solutions of the full system, a multi-time-scale perturbation method is used to resolve the interaction of solutions of the Schrödinger equations and a rapidly propagating shock wave. The leading order interaction equations are analyzed by the method of characteristics. The
details of the interaction process depend on the relative sizes of the shock strength and the dispersive wave amplitude. We show, for example, that if the shock strength is small compared to the dispersive wave amplitude, the shock can be completely blocked by the dispersive wave. Also, the dispersive wave can cause transient shocks to develop. The influence of the shock on the dispersive wave is manifested, to leading order, in the generalized frequency of the dispersive wave: the fast-time part of the frequency is the shock wave itself. Hence the frequency undergoes sudden jumps across the shock layers. Numerical experiments are presented which exemplify how, to leading order, the shock wave and dispersive wave frequency evolve in fast time.

## To Bettina Axel.

## Acknowledgments

I thank my advisor, Professor Paul Newton, for teaching me how to be a mathematician and for making the lesson so enjoyable. I thank my friend, James Colliander, for the inspiring mathematical conversations over the years. I thank my brothers, Jonathan and Howard, and my parents, Peter and Bettina, for everything.

## Contents

Introduction ..... 1
1 The weak coupling limit .....  6
1 The Riemann problem ..... 7
1.1 Case 1: $z \equiv 0$ ..... 8
1.2 Case 2: $z \neq 0$ ..... 10
1.3 Construction of periodic orbits ..... 12
2 The zero diffusion limit ..... 14
2.1 The general perturbation expansion ..... 16
2.2 Asymptotic approximation in region I ..... 23
2.3 Asymptotic approximation in region II ..... 25
2.4 Asymptotic approximation in region III ..... 27
3 Dynamics prior to shock onset ..... 28
3.1 The general procedure: variation of parameters. ..... 29
3.2 Derivation of solutions ..... 31
3.3 Transients of the Riemann problem ..... 33
3.4 Some phase-amplitude plots through break time ..... 35
2 The incompressible limit ..... 43
4 Exact solutions ..... 44
5 The $\epsilon=0$ problem ..... 47
6 The multi-scale expansions ..... 49
6.1 Leading order shock equations ..... 50
7 Modulation equations for the dispersive wave ..... 51
7.1 $O(1)$ modulation equations for Problem 1 ..... 51
7.2 $O$ (1) modulation equations for Problem 2 ..... 52
7.3 Generalized frequency of the dispersive wave ..... 53
7.4 $O(\epsilon)$ modulation equations for Problem 1 ..... 54
7.5 $O(\epsilon)$ modulation equations for Problem 2 ..... 55
7.6 Expansion summary ..... 56
8 Solution of the shock equation for Problem 1 ..... 57
8.1 Solution of the Riemann problem ..... 59
8.2 Shock structure for Problem 1 ..... 60
9 Solution of the shock equation for Problem 2 ..... 62
9.1 Characteristic equations ..... 64
9.2 Single or multiple shocks ..... 67
9.3 Long-time dynamics ..... 72
10 Numerical experiments ..... 79
Appendix ..... 96
References ..... 100
Vita ..... 105

## Introduction

The simplest equation which allows shock formation even for smooth initial data is the hyperbolic conservation law

$$
\begin{equation*}
u_{t}+u u_{x}=0 . \tag{0.1}
\end{equation*}
$$

It is a special case of the equation $u_{t}+f(u)_{x}=0$. Much work in the past fifty years has been devoted to the development of a general theory for this equation and its extension to systems of conservation laws [1, 2]. See also [3] for a very thorough list of significant contributions up to 1980. Dissipative or dispersive effects may be introduced in (0.1) by adding appropriate terms, typically higher order derivative terms or source terms, which may compete with the nonlinearity and suppress shock formation. The basic model of a shock wave in the presence of diffusion is Burgers' equation [4]

$$
\begin{equation*}
u_{t}+u u_{x}=\eta u_{x x} . \tag{0.2}
\end{equation*}
$$

The $\eta u_{x x}$ term smoothes out the solution and prohibits shock formation. For a large class of initial data, Burgers' equation is exactly solvable [5] via the Cole-Hopf transformation. Another method of introducing damping effects is seen in the model

$$
\begin{equation*}
u_{t}+u u_{x}=\sigma(x) u . \tag{0.3}
\end{equation*}
$$

The source term $\sigma u$ can act as a damping term [2] which in some cases may prevent shock formation [6]. This equation is studied in [7] as a model of gas flow through a nozzle. For some recent results on stability of the solutions of ( 0.3 ) when $\sigma$ is constant, see [8]. In [9, 10], more general nonlinearities are considered.

The fundamental and perhaps most well studied model which includes dispersive effects in the conservation law is the scalar KdV equation [2]

$$
\begin{equation*}
u_{t}+u u_{x}+u_{x x x}=0 \tag{0.4}
\end{equation*}
$$

The effect of the additional dispersive term, with its accompanying oscillations, has been well studied [ $11,12,13$ ]. The KdV equation is a rather special model of shockdispersion competition in that it has a completely integrable structure [14, 15]. For a
general discussion of various models of wave propagation through a dispersive environment, see [16, 17]. Equations which incorporate both diffusive and dispersive effects are also of interest $[18,19,20]$. These equations have the form

$$
\begin{equation*}
u_{t}+f(u)_{x}=\alpha u_{x x}+\beta u_{x x x} \tag{0.5}
\end{equation*}
$$

where, for example, $f(u)=u^{2}$ (KdV-Burgers) or $f(u)=u^{3}$ (modified KdV-Burgers).
Examples (0.2)-(0.5) are scalar equations which include nonlinear terms that favor shock formation and diffusive or dispersive terms which discourage shock formation. As such, they are limited in the sense that they describe a single mode subject to various competing forces (friction, advection, dispersion, etc.). More complicated models occur as systems of equations governing the dynamics of two or more modes interacting with each other in various ways. A simple example of such a system is a spring oscillating in the transverse direction (like a tight string) while compressive waves propagate along it in the longitudinal direction. To describe certain long wave/short wave resonant interactions in shallow water, Djordjevic \& Redekopp [21] derived the system

$$
\begin{align*}
& u_{t}+\alpha\left(|E|^{2}\right)_{x}=0  \tag{0.6a}\\
& i E_{t}+E_{x x}-u E=0 . \tag{0.6b}
\end{align*}
$$

Here, the real part of $E(x, t)$ represents a short 'compressive' wave (a capillary wave due to water surface tension) and the real-valued $u(x, t)$ represents a long wave oscillating in the transverse direction (a water wave due to gravity). The governing equation ( 0.6 b ) for $E$ is the dispersive Schrödinger equation with potential $u$. In [22], this system is solved via the inverse scattering transform developed by Ablowitz et al. [23]. A similar pair of equations originating from plasma turbulence theory is the so-called Zakharov system [24]

$$
\begin{align*}
& \epsilon^{2} u_{t t}-\left(u+|E|^{2}\right)_{x x}=0  \tag{0.7a}\\
& i E_{t}+E_{x x}-u E=0, \tag{0.7b}
\end{align*}
$$

coupling an ion-sound wave $u$ with a dispersive plasma envelope $E$. Although the coupling is nonlinear, the acoustic wave $u$ is governed by a linear wave equation. Therefore,
unless shocks are introduced in the acoustic wave as initial data, they will not develop. Gibbons et al. show in [25] that (0.7) has a canonical Hamiltonian structure, but it is not known if the system is completely integrable. As the parameter $\epsilon$ goes to zero in (0.7a), the speed of the acoustic wave goes to infinity. This limit is therefore called the incompressible limit. A general program for studying incompressible limits has been initiated by Ebin [26] and Klainerman \& Majda [27]. For small $\epsilon$, (0.7) has two widely separated time scales: $t$ and $t / \epsilon$. Letting $\epsilon \rightarrow 0$ in (0.7a) gives $u=-|E|^{2}$ (modulo a function of $t$, and substituting this into ( 0.7 b ) shows that when $\epsilon \rightarrow 0, E$ satisfies

$$
\begin{equation*}
i E_{t}+E_{x x}+|E|^{2} E=0 \tag{0.8}
\end{equation*}
$$

the completely integrable cubic nonlinear Schrödinger equation (NLS). See [14, 28, 29] for a discussion of NLS. The incompressible limit is thus an integrable limit: for small $\epsilon$, system ( 0.7 ) is a perturbation of NLS, but the perturbation term $u$ is a fast wave, not a small magnitude wave. This limit is studied by Schochet \& Weinstein in [30], by Ozawa \& Tsutsumi in [31], and by Newton in [32]. In [32], the incompressible limit is investigated by a multi-time-scale perturbation method which takes advantage of the widely separated time scales when $\epsilon$ is small. The method is used to analyze the interactions of a fast unidirectional acoustic plane wave and a soliton solution of the nonlinear Schrödinger equation.

In this doctoral thesis, a similar program is carried out for a system based on the Zakharov system, but one which allows the development of shocks in the acoustic wave u. Mimicking the Zakharov system, the model we consider here is

$$
\begin{align*}
& \epsilon u_{t}+u\left(u+\delta|E|^{2}\right)_{x}=0  \tag{0.9a}\\
& i E_{t}+E_{x x}-u E=0 . \tag{0.9b}
\end{align*}
$$

Although not explicitly derived from physical principals, it is a natural generalization of the Zakharov and Djordjevic-Redekopp systems in that it replaces the acoustic wave equations in those systems by the simplest possible nonlinear equation which allows shock formation. This is seen by comparing the three systems in the absence of a dispersive wave envelope ( $E=0$ ). The Djordjevic-Redekopp system reduces to
$u_{t}=0$, the Zakharov system reduces to $\epsilon^{2} u_{t t}-u_{x x}=0$, while system ( 0.9 ) reduces to $\epsilon u_{t}+u u_{x}=0$, hence shock formation is admitted. System (0.9) is designed to incorporate the interaction of shock waves and dispersive waves while not straying too far (structurally) from these well-known physical models. In particular, the equation governing the dispersive wave envelope $E$ is exactly the same in all three systems. We note some immediate observations regarding (0.9).

- Like the Zakharov system, there is no known integrable structure for system (0.9).
- When $\delta=0$, the equations are coupled only by the presence of $u(x, t)$ in $(0.9 \mathrm{~b})$ as the time dependent potential. We call this the weak coupling limit; it is the subject of Chapter 1 of this thesis ${ }^{1}$.
- When there is no dispersive wave ( $E(x, t)=0$ ), the acoustic wave is governed by the scalar conservation law

$$
\begin{equation*}
\epsilon u_{t}+u u_{x}=0, \tag{0.10}
\end{equation*}
$$

whose wavespeed is inversely proportional to $\epsilon$. In the singular limit $\epsilon \rightarrow 0$, equation ( 0.9 a) can be solved by setting $u=-\delta|E|^{2}$. When this is substituted into ( 0.9 b ), we get the NLS equation for $E(x, t)$. Hence, this limit, as in the Zakharov system, is an integrable limit. It is an incompressible limit in the sense that the wavespeed $u / \epsilon \rightarrow \infty$. The incompressible limit of the fully coupled ( $\delta=1$ ) system, is the subject of Chapter 2 of this thesis.

Despite the fact that, to our knowledge, this is the first comprehensive study of two mode shock-dispersive wave interactions, the propagation of shock waves through a dispersive environment occurs in many natural settings. One example comes about when acoustic shocks (e.g., underwater explosions) propagate through the ocean where they potentially can interact with surface gravity waves or internal waves [34, 35]. For general background on these and related issues, see [36]. A second example arises when a supersonic object travels over the ocean, generating a sonic boom and leaving

[^0]a footprint on the water's surface. The interaction of this N -wave shock on the flat ocean surface is examined in $[37,38,39]$. The more complicated situation in which deep ocean waves interact with the $N$-wave has not been studied and is a clear example of the interaction of shocks with finite amplitude dispersive waves. Another example arises in medical applications where acoustic waves propagate through tissue and intemal organs and are used for non-destructive imaging, or as destructive devices, for example, to break up kidney stones. A basic introduction to some of these issues can be found in [40]. See also [41, 42]. As a last example, we mention the propagation of shock waves through interstellar plasma media [43, 44, 45]. This may result from energetic events such as mass loss by massive stars, supernova explosions, and cloud-cloud collisions, for example. Such wave interactions can trigger a weak form of plasma turbulence [44] and are thought to periodically cause interruptions in satellite based communication systems on earth. The results of this thesis are intended as a first step in describing such complex interaction problems.

## Chapter 1

## The weak coupling limit

The weak coupling limit is obtained by letting $\delta \rightarrow 0$ in (0.9a). This gives

$$
\begin{align*}
& u_{t}+u u_{x}=0  \tag{1.0a}\\
& i E_{t}+E_{x x}-u E=0 . \tag{1.0b}
\end{align*}
$$

To solve (1.0), one must specify initial data $u(x, 0)=u_{0}(x), E(x, 0)=E_{0}(x)$ as well as boundary conditions as $|x| \rightarrow \infty$. The main question we are concerned with is how the formation of shocks in the conservation law (1.0a) affects the dispersive wave envelope governed by (1.0b).

Chapter 1 is organized as follows. In section 1 we consider the Riemann problem for (1.0a) in which an acoustic shock propagates at a fixed constant speed determined by the initial data. We solve for the dispersive wave envelope $E(x, t)=R(x, t) \exp (i \theta(x, t))$ by moving in a frame of reference with the shock and solving the resulting coupled ode's for the amplitude $R$ and phase $\theta$. In section 2 we replace (1.0a) with the Burgers' equation and study the zero diffusion limit. We construct small diffusion asymptotic expansions for the amplitude $R$ and show that solutions constructed in section 1 are zero diffusion limits of the augmented problem. For the asymptotic expansion, it is necessary to keep track of algebraically small as well as exponentially small terms. The exponentially small terms play an important role in enforcing continuity of the amplitude in the region of rapid transition of the smoothed shock profile of Burgers' equation. In section 3, we specify initial data for $u(x, t)$ which evolves to a shock in finite time and study the resulting time dependent dispersive wave. It is shown that the formation of the shock drives a blow-up in the phase gradient, $\theta_{x}$, of the dispersive wave: the shock for $u(x, t)$ develops algebraically in time, the phase gradient blows up logarithmically in time. We then show that a subset of these time dependent solutions
are transients of the Riemann problem steady states of section 1. In section 3.3, we show plots of three time dependent solutions. The first is a solution that evolves to a steady state constructed in section 1. The second is a solution that evolves to a phase-kinked steady state, i.e. a traveling wave with a phase discontinuity. The third is a non-transient solution.

## 1 The Riemann problem

We first consider the Riemann problem for system (1.0). Given initial data

$$
u(x, 0)= \begin{cases}u_{l} & x<0  \tag{1.1}\\ u_{r} & x>0\end{cases}
$$

where $u(x, t): \mathbf{R} \times \mathbf{R}^{+} \rightarrow \mathbf{R}, u_{l}-u_{r}>0$, we seek solutions, $E(x, t): \mathbf{R} \times \mathbf{R}^{+} \rightarrow \mathbf{C}$, of the time dependent Schrödinger equation (1.0b).

The solution to the conservation law (1.0a) with initial data (1.1) is given by a propagating shock with speed $c=\frac{1}{2}\left(u_{l}+u_{r}\right),[1,46]$. We assume without loss of generality that $u_{l}+u_{r}>0$. In terms of the traveling variable $\xi=x-c t$, the shock solution is

$$
u(\xi)= \begin{cases}u_{l} & \xi<0  \tag{1.2}\\ u_{\mathrm{r}} & \xi>0\end{cases}
$$

We seek solutions to the time dependent Schrödinger equation that move with the same speed as the shock, hence are steady in the traveling variable. Substituting $E(\xi)=R(\xi) \exp (i \theta(\xi))$ into (1.0b) leads to the coupled ordinary differential equations

$$
\begin{align*}
& \theta^{\prime \prime} R+\left(2 \theta^{\prime}-c\right) R^{\prime}=0  \tag{1.3a}\\
& R^{\prime \prime}+\left(c \theta^{\prime}-\left(\theta^{\prime}\right)^{2}-u\right) R=0, \tag{1.3b}
\end{align*}
$$

where prime is differentiation with respect to $\xi$. Since only derivatives of $\theta$ appear, it is convenient to define a new variable

$$
\begin{equation*}
z(\xi)=2 \theta^{\prime}-c . \tag{1.4}
\end{equation*}
$$

We write

$$
R(\xi)=\left\{\begin{array}{ll}
R_{l}(\xi) & \xi<0 \\
R_{r}(\xi) & \xi>0,
\end{array} \quad \theta(\xi)=\left\{\begin{array}{ll}
\theta_{l}(\xi) & \xi<0 \\
\theta_{r}(\xi) & \xi>0,
\end{array} \quad z(\xi)= \begin{cases}z_{l}(\xi) & \xi<0 \\
z_{r}(\xi) & \xi>0\end{cases}\right.\right.
$$

and seek $R_{i}(\xi)$ and $\theta_{i}(\xi), i=l, r$. We then obtain solutions of (1.3) by piecing together the solutions for $\xi<0$ and $\xi>0$ subject to the continuity condition

$$
R_{l}\left(0^{-}\right)=R_{r}\left(0^{+}\right) \quad \text { and } \quad \theta_{l}\left(0^{-}\right)=\theta_{r}\left(0^{+}\right) .
$$

We consider two cases.

### 1.1 Case 1: $z \equiv 0$

In this case, both parts of the phase, $\theta_{l}$ and $\theta_{r}$, are linear in $\xi$. Equation (1.4) implies

$$
\theta(\xi)= \begin{cases}c \xi / 2+\gamma^{\prime} & \xi<0  \tag{1.5}\\ c \xi / 2+\gamma_{r} & \xi>0\end{cases}
$$

The constants $\gamma_{l}$ and $\gamma_{r}$ are defined modulo multiples of $2 \pi$. In the case $\gamma_{l} \neq \gamma_{r}(\bmod$ $2 \pi$ ), (1.5) represents a steady state phase-kink, i.e, a phase with a jump discontinuity across $\xi=0$. If $\gamma_{l}=\gamma_{r}(\bmod 2 \pi)$, the phase is continuous. Equation (1.3b) reduces to the time-independent Schrödinger equation for the amplitudes $R_{l}$ and $R_{r}$ :

$$
\begin{array}{ll}
R_{l}^{\prime \prime}+V_{l} R_{l}=0 & \xi<0 \\
R_{r}^{\prime \prime}+V_{r} R_{r}=0 & \xi>0, \tag{1.6b}
\end{array}
$$

where $V_{l r}=\frac{c^{2}}{4}-u_{l r}$. The bounded solutions of (1.6) depend on the signs of $V_{l}$ and $V_{r}$. Figure 1 displays the regions of the $\left(u_{r}, u_{l}\right)$ plane in which $V_{l}$ and $V_{r}$ have definite signs. Because $u_{l} \pm u_{r}>0$, there are only three allowable subregions of shock values. Region I corresponds to $V_{l}<0, V_{r} \leq 0$, region II to $V_{l} \leq 0, V_{r}>0$, while region III corresponds to $V_{l}>0, V_{r}>0$. The dashed line in Figure 1 is the line $u_{l}=u_{r}$. The bounded continuous solutions in the three regions are


Figure 1: The ( $u_{r}, u_{l}$ ) plane consisting of allowable shock strengths on either side of $\xi=0$. Region I: amplitude $R$ decays exponentially on each side of $\xi=0$. Region II: $R$ decays on the left, oscillates on the right. Region III: $R$ oscillates on both sides. All regions are above the dashed line $u_{l}=u_{r}$.
region I:

$$
R(\xi)= \begin{cases}\alpha \exp \left(\sqrt{-V_{l} \xi}\right) & \xi<0  \tag{1.7a}\\ \alpha \exp \left(-\sqrt{-V_{\mathrm{r}}}\right) & \xi>0\end{cases}
$$

region II:

$$
R(\xi)= \begin{cases}\alpha \exp \left(\sqrt{\left.-V_{l} \xi\right)}\right. & \xi<0  \tag{1.7b}\\ \sqrt{\frac{-V_{l}}{V_{r}}} \alpha \sin \left(\sqrt{V_{r}} \xi\right)+\alpha \cos \left(\sqrt{V_{r}} \xi\right) & \xi>0\end{cases}
$$

region III:

$$
R(\xi)= \begin{cases}\frac{1}{\nabla_{l}} \beta \sin \left(\sqrt{V_{l}} \xi\right)+\alpha \cos \left(\sqrt{V_{l}} \xi\right) & \xi<0  \tag{1.7c}\\ \frac{1}{\nabla_{r}} \beta \sin \left(\sqrt{V_{r}} \xi\right)+\alpha \cos \left(\sqrt{V_{r}} \xi\right) & \xi>0\end{cases}
$$

The constants $\alpha$ and $\beta$ are determined by $R(0)$ and $R^{\prime}(0)$, respectively.

### 1.2 Case 2: $z \neq 0$

Solving (1.4) for $\theta(\xi)$ gives

$$
\theta(\xi)= \begin{cases}c \xi / 2+\psi_{l}(\xi)+\gamma_{l} & \xi<0  \tag{1.8}\\ c \xi / 2+\psi_{r}(\xi)+\gamma_{r} & \xi>0\end{cases}
$$

where

$$
\begin{equation*}
\psi_{i}(\xi)=\frac{1}{2} \int_{0}^{\xi} z_{i}(s) d s, \quad i=l, r, \tag{1.9}
\end{equation*}
$$

and $\gamma_{l}, \gamma_{r}$ are constants. In the limit $z_{i} \rightarrow 0,(1.8)$ reduces to the linear phase (1.5). With $\theta$ given by (1.8), equation (1.3a) separates as

$$
\begin{equation*}
-\frac{z_{i}^{\prime}}{2 z_{i}}=\frac{R_{i}^{\prime}}{R_{i}}, \tag{1.10}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
R_{i}(\xi)=\frac{K_{i}}{\sqrt{\left|z_{i}(\xi)\right|}}, \quad K_{i} \text { constant } . \tag{1.11}
\end{equation*}
$$

Substituting (1.11) into (1.3b) gives a nonlinear oscillator equation for $z_{i}$ :

$$
\begin{equation*}
z_{i}^{\prime \prime}+f\left(z_{i}, z_{i}^{\prime}\right)+g\left(z_{i}\right)=0, \tag{1.12}
\end{equation*}
$$

where

$$
\begin{equation*}
f\left(z_{i}, z_{i}^{\prime}\right)=-\frac{3}{2}\left(z_{i}^{\prime}\right)^{2} z_{i}^{-1}, \quad g\left(z_{i}\right)=\frac{1}{2} z_{i}^{3}-2 V_{i} z_{i} . \tag{1.13}
\end{equation*}
$$

To analyze (1.12) in more detail, we write it in first order form:

$$
\begin{align*}
& z_{i}^{\prime}=w_{i}  \tag{1.14a}\\
& w_{i}^{\prime}=\frac{3}{2} \frac{w_{i}^{2}}{z_{i}}+2 V_{i} z_{i}-\frac{1}{2} z_{i}^{3} \tag{1.14b}
\end{align*}
$$

If $V_{i}>0, i=l, r$ (region III in Figure 1), then (1.14) has the fixed points

$$
\begin{align*}
& \bar{z}_{i}= \pm 2 \sqrt{V_{i}}  \tag{1.15}\\
& \bar{w}_{i}=0 . \tag{1.16}
\end{align*}
$$

The fixed points correspond to solutions

$$
\begin{align*}
& R= \begin{cases}\frac{K_{l}}{\sqrt{2}} V_{l}^{-\frac{1}{4}} & \xi<0 \\
\frac{K_{r}}{\sqrt{2}} V_{r}^{-\frac{1}{4}} & \xi>0,\end{cases}  \tag{1.17}\\
& \theta= \begin{cases}\left(c / 2 \pm \sqrt{V_{l}}\right) \xi+\gamma_{l} & \xi<0 \\
\left(c / 2 \pm \sqrt{V_{r}}\right) \xi+\gamma_{r} & \xi>0,\end{cases} \tag{1.18}
\end{align*}
$$

of system (1.3), where $K_{l}, K_{r}, \gamma_{l}, \gamma_{r}$ are constants. We show that these fixed point solutions are centers, hence in the neighborhood of the fixed points there exist periodic solutions.

Without loss of generality, we choose the plus sign in (1.15). A suitable transformation of the fixed point and a rescaling of the dependent and independent variables renders system (1.14) in normal form. Let $z_{i}=\bar{z}_{i}+q_{1} / \bar{z}_{i}$ and define $\hat{\xi}=\tilde{z}_{j} \xi$ and $q_{2}=d q_{1} / d \hat{\xi}$. With $\cdot=d / d \hat{\xi}$ and $q=\left(q_{1}, q_{2}\right),(1.14)$ becomes

$$
\begin{equation*}
\dot{q}=A q+f(q) \tag{1.19}
\end{equation*}
$$

where

$$
A=\left(\begin{array}{cc}
0 & 1  \tag{1.20}\\
-1 & 0
\end{array}\right), \quad f(q)=\binom{f_{1}(q)}{f_{2}(q)}
$$

and

$$
\begin{align*}
& f_{1}(q)=0,  \tag{1.21}\\
& f_{2}(q)=\frac{3}{2}\left[\frac{q_{2}^{2}}{\bar{z}_{i}^{2}+q_{1}}-\frac{q_{1}^{2}}{z_{i}^{2}}-\frac{q_{1}^{3}}{3 \bar{z}_{i}^{4}}\right] . \tag{1.22}
\end{align*}
$$

We make three observations about system (1.19).

1. The linear part of (1.19) has pure imaginary eigenvalues $\lambda= \pm i$, hence no conclusion can be drawn on the nature of the fixed point $q=0$ based on linear theory.
2. $f(q)$ is analytic at the origin.
3. (1.19) is symmetric under time reversal, i.e., if $\left(q_{1}(\hat{\xi}), q_{2}(\hat{\xi})\right.$ ) is a solution, so is $\left(q_{1}(-\hat{\xi}),-q_{2}(-\hat{\xi})\right)$.

By the Center Theorem of Liapunov [47], the first two observations allow us to conclude that the equilibrium point $q=0$ is either unstable, asymptotically stable, or a center. The third observation implies that the fixed point $q=0$ is a center.

The periodic solutions for $q$ near 0 correspond to solutions of (1.3) that have oscillatory amplitudes. Although these amplitudes are qualitatively similar to the solutions discussed in the $z \equiv 0$ case, they are not simple sinusoidal oscillations. In the next section we construct an approximation of these periodic orbits.

### 1.3 Construction of periodic orbits

We approximate the periodic solutions for $z_{i}(\xi)$ near the fixed points $\bar{z}_{i}$ defined in (1.15), i.e., near $q=0$. For this approximation, we transform to polar coordinates and follow the general method described in [47]. Let $q_{1}=r \cos (\theta)$ and $q_{2}=-r \sin (\theta)$. Then system (1.19) becomes

$$
\begin{equation*}
\dot{r}=R(r, \theta), \quad \dot{\theta}=1+\theta(r, \theta) \tag{1.23}
\end{equation*}
$$

The right-hand sides are given by

$$
\begin{gather*}
R(r, \theta)=f_{1}(r \cos \theta,-r \sin \theta) \cos \theta-f_{2}(r \cos \theta,-r \sin \theta) \sin \theta  \tag{1.24}\\
\theta(r, \theta)=-\frac{1}{r} f_{1}(r \cos \theta,-r \sin \theta) \sin \theta-f_{2}(r \cos \theta,-r \sin \theta) \cos \theta \tag{1.25}
\end{gather*}
$$

where $\Theta(0, \theta)$ is defined to be 0 . Both $R(r, \theta)$ and $\theta(r, \theta)$ are $2 \pi$-periodic functions of $\theta$, and in a neighborhood of $q=0$ we have $\dot{\theta}>0$. Dividing the first equation by the second in (1.23) gives a scalar ordinary differential equation governing the radial variable:

$$
\begin{equation*}
\frac{d r}{d \theta}=\hat{R}(r, \theta) \tag{1.26}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{R}(r, \theta)=\frac{R(r, \theta)}{1+\theta(r, \theta)} \tag{1.27}
\end{equation*}
$$

By (1.21), (1.22), equations (1.24), (1.25) are

$$
\begin{align*}
& R(r, \theta)=-\frac{3}{2} \sin (\theta)\left[\frac{r^{2} \sin ^{2}(\theta)}{\bar{z}_{j}^{2}+r \cos (\theta)}-\frac{r^{2} \cos ^{2}(\theta)}{\bar{z}_{j}^{2}}-\frac{r^{3} \cos ^{3}(\theta)}{3 \bar{z}_{j}^{4}}\right],  \tag{1.28}\\
& \Theta(r, \theta)=-\frac{3}{2} \cos (\theta)\left[\frac{r^{2} \sin ^{2}(\theta)}{\bar{z}_{j}^{2}+r \cos (\theta)}-\frac{r^{2} \cos ^{2}(\theta)}{\bar{z}_{j}^{2}}-\frac{r^{3} \cos ^{3}(\theta)}{3 \bar{z}_{j}^{4}}\right] . \tag{1.29}
\end{align*}
$$

Expanding the right-hand sides of (1.28), (1.29) in terms of $r$ for $r$ near the origin and substituting into (1.27) gives

$$
\begin{equation*}
\hat{R}(r, \theta)=-\alpha \sin (\theta) r^{2}+O\left(r^{3}\right) \tag{1.30}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=\frac{3}{2} \frac{\sin ^{2}(\theta)}{\bar{z}_{j}^{2}}-\frac{3}{2} \frac{\cos ^{2}(\theta)}{\bar{z}_{j}^{2}} \tag{1.31}
\end{equation*}
$$

We transform the radial variable in (1.26) to eliminate coefficients depending on $\theta$. Let

$$
\begin{equation*}
r=\rho+\beta(\theta) \rho^{2}+O\left(\rho^{3}\right) \tag{1.32}
\end{equation*}
$$

where $\beta(\theta)$ is to be determined. Then

$$
\begin{equation*}
\frac{d r}{d \theta}=\frac{d r}{d \rho} \frac{d \rho}{d \theta}=(1+2 \beta \rho) \frac{d \rho}{d \theta}+\beta^{\prime} \rho^{2} \tag{1.33}
\end{equation*}
$$

Using this in (1.26) gives the new radial equation

$$
\begin{equation*}
\frac{d \rho}{d \theta}=-\left(\alpha \sin (\theta)+\beta^{\prime}\right) \rho^{2}+O\left(\rho^{3}\right) \tag{1.34}
\end{equation*}
$$

For $\rho$ to be independent of $\theta$ through $O\left(\rho^{2}\right), \beta(\theta)$ must satisfy:

$$
\begin{equation*}
\beta^{\prime}=-\alpha \sin (\theta) \tag{1.35}
\end{equation*}
$$

Integrating gives

$$
\begin{equation*}
\beta(\theta)=\frac{3}{2 \bar{z}_{j}^{2}} \cos (\theta)-\frac{1}{\bar{z}_{j}^{2}} \cos ^{3}(\theta)-\frac{1}{2 \bar{z}_{j}^{2}} \tag{1.36}
\end{equation*}
$$

where the integration constant was chosen so that $\beta(0)=0$. Hence, through $O\left(\rho^{2}\right)$, we have the approximate periodic orbit given by (1.32), (1.36) with $\rho=$ constant. Note that in (1.36), the $\tilde{z}_{j}^{2}$ term in the denominators is $4 V_{l}$ if $\xi<0$ and $4 V_{r}$ if $\xi>0$.


Figure 2: Smooth shock layer initial data (2.2) for Burgers' equation (2.1a).

## 2 The zero diffusion limit

For pure conservation laws, the zero diffusion limiting process picks out a unique and physically meaningful shock solution in what is otherwise a problem with many solutions [ $1,46,48,49,50$ ]. In this section we show that the solutions to the Riemann problem (1.0), (1.1) constructed previously are zero diffusion limits of an augmented model that incorporates the effects of viscosity. The augmented model for $u^{\epsilon}(x, t ; \epsilon)$ and $E^{\epsilon}(x, t ; \epsilon)$ is

$$
\begin{align*}
& u_{t}^{\epsilon}+u^{\epsilon} u_{x}^{\epsilon}=\epsilon u_{x x}^{\epsilon}  \tag{2.1a}\\
& i E_{t}^{\epsilon}+E_{x x}^{\epsilon}-u^{\epsilon} E^{\epsilon}=0, \tag{2.1b}
\end{align*}
$$

where $0<\epsilon \ll 1$. As initial data for Burgers' equation (2.1a) we take the smoothed profile (Figure 2) corresponding to (1.1):

$$
\begin{equation*}
u^{\epsilon}(x, 0 ; \epsilon)=u_{r}+\frac{\left(u_{l}-u_{r}\right)}{1+\exp (a x / \epsilon)}, \tag{2.2}
\end{equation*}
$$

where $a=\frac{1}{2}\left(u_{l}-u_{r}\right)>0$. In the limit $\epsilon \rightarrow 0$, for $x \neq 0$, it is easy to verify that $u^{\epsilon}(x, 0 ; \epsilon) \rightarrow u(x, 0)$ given in (1.1). The solution to (2.1a) with initial data (2.2) can be explicitly written [2]:

$$
\begin{equation*}
u^{\epsilon}(\xi ; \epsilon)=u_{r}+\frac{\left(u_{l}-u_{r}\right)}{1+\exp (a \xi / \epsilon)}, \tag{2.3}
\end{equation*}
$$

where the traveling variable $\xi$ and wavespeed $c$ are those given in the previous section. We construct explicit asymptotic expansions for $E^{\epsilon}$ in regions I, II, III which, in the limit $\epsilon \rightarrow 0$, reduce to (1.5), (1.7) of section 1.1. The asymptotic expansions for $R^{\epsilon}(\xi)$ are summarized first for convenience.

Theorem 1 (Small diffusion expansions). Consider solutions to (2.1b) of the form $E^{\epsilon}(\xi ; \epsilon)=R^{\epsilon}(\xi ; \epsilon) \exp \left(i \theta^{\epsilon}(\xi ; \epsilon)\right)$, with $\xi=x-c t, c=\left(u_{l}+u_{\top}\right) / 2$. If the phase $\theta^{\epsilon}(\xi ; \epsilon)$ is independent of $\epsilon$ and corresponds to the limiting $(\epsilon=0)$ form in equation (1.5), then the amplitude $R^{\epsilon}(\xi)$ satisfies the Schrödinger equation

$$
\begin{equation*}
\frac{d^{2} R^{\epsilon}}{d \xi^{2}}+\left(\frac{c^{2}}{4}-u^{\epsilon}(\xi)\right) R^{\epsilon}=0 \tag{2.4}
\end{equation*}
$$

where $u^{\epsilon}(\xi)$ is given in (2.3). The approximate solutions through $O\left(\epsilon^{2}\right)$ are:
region I:

$$
R^{\epsilon}(\xi)= \begin{cases}\left\{\alpha+\frac{\alpha \pi^{2}}{6 a} \epsilon^{2}+O\left(\epsilon^{3}\right)-\frac{2 \alpha \epsilon^{2}}{a} \exp \left(\frac{\xi}{\epsilon}\right)+t . s . t\right\} \exp \left(\sqrt{-V_{l}} \xi\right) & \xi<0,  \tag{2.5a}\\ \left\{\alpha-\frac{\alpha \pi^{2}}{6 a} \epsilon^{2}+O\left(\epsilon^{3}\right)+\frac{2 \alpha \epsilon^{2}}{a} \exp \left(\frac{-\xi}{\epsilon}\right)+t . s . t\right\} \exp \left(-\sqrt{-V_{r}} \xi\right) & \xi>0,\end{cases}
$$

region II:
region III:
where t.s.t. denotes terms which, for fixed $\xi$, are transcendentally small as $\epsilon \rightarrow 0$.
In the limit $\epsilon \rightarrow 0$, these solutions correspond to the solutions (1.5), (1.7) of the zero diffusion model. Note that the terms in brackets are either algebraic in $\epsilon$ or transcendentally small. We have included the first transcendentally small term in each case, as it represents the most important contribution to the solution in the inner region $|\xi| \ll 1$. In the terminology of matched asymptotics [51], the algebraically small terms constitute the outer solution while the transcendentally small terms constitute the inner solution. The composite expansions (2.5) are obtained by adding the inner and outer contributions. The remainder of this section is devoted to constructing these asymptotic expansions. In Figures 3-5, plots of the expansions are presented. Each figure contains three plots: the first two plots show the outer and composite expansions in each region, the third plot compares the composite expansion to a numerical solution of equation (2.4).

### 2.1 The general perturbation expansion

Our analysis is based on system (1.3) with potential $u^{\epsilon}(\xi)$ given by (2.3). With $z=$ $2 \theta^{\prime}-c=0$, the phase $\theta$ is given by (1.5). It is then straightforward to show that (1.3b)


Figure 3: Region I ( $u_{l}=2, u_{r}=1, \alpha=1$ ). (a) Outer expansion. Solid: $\epsilon=0$. Dashed: $\epsilon=0.3$. (b) Composite expansion (2.5a). Solid: $\epsilon=0$. Dashed: $\epsilon=0.3$. (c) Composite expansion (2.5a) (dashed) vs. numerical solution of equation (2.4) (solid) each with $\epsilon=0.3$.


Figure 4: Region II ( $u_{l}=6, u_{r}=2, \alpha=1$ ). (a) Outer expansion. Solid: $\epsilon=0$. Dashed: $\epsilon=0.3$. (b) Composite expansion (2.5b). Solid: $\epsilon=0$. Dashed: $\epsilon=0.3$. (c) Composite expansion (2.5b) (dashed) vs. numerical solution of equation (2.4) (solid) each with $\epsilon=0.3$.


Figure 5: Region III ( $u_{l}=6, u_{r}=4, \alpha=1, \beta=1$ ). (a) Outer expansion. Solid: $\epsilon=0$. Dashed: $\epsilon=0.3$. (b) Composite expansion (2.5c). Solid: $\epsilon=0$. Dashed: $\epsilon=0.3$. (c) Composite expansion (2.5c) (dashed) vs. numerical solution of equation (2.4) (solid) each with $\epsilon=0.3$.
reduces to the Schrödinger equation (2.4). We seek solutions of (2.4) which have the same form as (1.7) but with coefficients $\alpha$ and $\beta$ replaced by functions $A, B, C, D$ of $\xi$ and $\epsilon$. Since the potential $V^{\epsilon}(\xi)=\frac{c^{2}}{4}-u^{\epsilon}(\xi)$ can be expanded as

$$
V^{\epsilon} \simeq \begin{cases}V_{l}+2 a \sum_{n=1}^{\infty}(-1)^{n+1} \exp (n a \xi / \epsilon) & \xi<0  \tag{2.6}\\ V_{r}-2 a \sum_{n=1}^{\infty}(-1)^{n+1} \exp (-n a \xi / \epsilon) & \xi>0\end{cases}
$$

where $V_{l r}=\frac{c^{2}}{4}-u_{l r}$, we are led to seek similar expansions for the coefficients $A, B, C$, D. Expansion (2.6) displays $V^{\epsilon}$ as the sum

$$
\begin{equation*}
V^{\epsilon}=\operatorname{Out}\left(V^{\epsilon}\right)+\operatorname{Inn}\left(V^{\epsilon}\right), \tag{2.7}
\end{equation*}
$$

where

$$
\operatorname{Out}\left(V^{\epsilon}\right)= \begin{cases}V_{l} & \xi<0  \tag{2.8}\\ V_{r} & \xi>0\end{cases}
$$

and

$$
\operatorname{Inn}\left(V^{\epsilon}\right)=\left\{\begin{array}{cl}
2 a \sum_{n=1}^{\infty}(-1)^{n+1} \exp \left(\frac{n a \xi}{\epsilon}\right) & \xi<0  \tag{2.9}\\
-2 a \sum_{n=1}^{\infty}(-1)^{n+1} \exp \left(\frac{-n a \xi}{\epsilon}\right) & \xi>0 .
\end{array}\right.
$$

The outer part of the expansion, $\operatorname{Out}\left(V^{\epsilon}\right)$, is a step function. The inner contribution, $\operatorname{Inn}\left(V^{\epsilon}\right)$, depends only on the scaled variable $\xi / \epsilon$, and for $\xi \neq 0$, is transcendentally small as $\epsilon \rightarrow 0$. As $\xi \rightarrow 0$, however, $\operatorname{Inn}\left(V^{\epsilon}\right)$ is of the same order of magnitude as Out $\left(V^{\epsilon}\right)$. The expansions for $A, B, C$, and $D$ will also be sums of outer and inner parts. These expansions will be subjected to matching conditions at $\xi=0$. The matching conditions will ensure that $R^{\epsilon}(0)=\alpha$, and that in regions II and III, $\frac{d R^{e}}{d \xi}(0)=\sqrt{-V_{l}} \alpha$ and $\frac{d R^{2}}{\mathbb{\xi}}(0)=\beta$, respectively. We define several operators that are needed to describe the solutions.

Definition: For $n \geq 1$, define the operators

$$
T_{n}^{r}=\frac{2 \epsilon^{3}}{n\left[(n a)^{2}+4 V_{r} \epsilon^{2}\right]}\left(\begin{array}{cc}
\frac{n a}{\epsilon} & -2 \sqrt{V_{r}}  \tag{2.10}\\
2 \sqrt{V_{r}} & \frac{n a}{\epsilon}
\end{array}\right)
$$

$$
\begin{gather*}
T_{n}^{l}=\frac{-2 \epsilon^{3}}{n\left[(n a)^{2}+4 V_{l} \epsilon^{2}\right]}\left(\begin{array}{cc}
\frac{n a}{\epsilon} & 2 \sqrt{V_{l}} \\
-2 \sqrt{V_{l}} & \frac{n a}{\epsilon}
\end{array}\right),  \tag{2.11}\\
S_{n}^{r}=\frac{2 \epsilon^{2}}{n\left(n a+2 \sqrt{\left.-V_{r} \epsilon\right)}\right.},  \tag{2.12}\\
S_{n}^{l}=\frac{-2 \epsilon^{2}}{n\left(n a+2 \sqrt{\left.-V_{l} \epsilon\right)}\right.} . \tag{2.13}
\end{gather*}
$$

Definition: Let $\delta=\left(\delta_{1}, \ldots, \delta_{j}\right), j \geq 1$, be a multiindex with positive, integral entries. We say that $\delta$ has length $j$. Define $T_{\delta}^{l}$ by

$$
\begin{equation*}
T_{\delta}^{l}=T_{\delta_{1}}^{l} \cdot T_{\delta_{2}}^{d} \cdots T_{\delta_{j}}^{l}, \tag{2.14}
\end{equation*}
$$

where the product is interpreted as a composition of operators. Define $T_{\delta}^{r}, S_{\delta}^{l}$, and $S_{\delta}^{r}$ similarly. Let $\Delta_{j}$ be that set of all multiindices of length $j$. Define

$$
\begin{equation*}
\Delta_{j}^{k}=\left\{\delta \in \Delta_{j}: \delta_{1}=k, \delta_{1}>\delta_{2}>\cdots>\delta_{j}\right\} \tag{2.15}
\end{equation*}
$$

Definition: Define the operators

$$
\begin{align*}
& p_{0}\left(T^{l}\right)=I, \text { the } 2 \times 2 \text { identity matrix, }  \tag{2.16}\\
& p_{n}\left(T^{d}\right)=\sum_{j=1}^{n}\left((-1)^{n-j} \sum_{\delta \in \Delta_{j}^{n}} T_{\delta}^{d}\right) n \geq 1 . \tag{2.17}
\end{align*}
$$

Define $p_{n}\left(T^{\top}\right), p_{n}\left(S^{l}\right)$, and $p_{n}\left(S^{\top}\right)$ similarly.

We shall need the following lemma and corollary.
Lemma 1. Let $\pi_{i}=R^{2} \rightarrow R$ be the projection operators:

$$
\pi_{i}\binom{x}{y}= \begin{cases}x & \text { if } i=1 \\ y & \text { if } i=2\end{cases}
$$

We have the following expansions:

$$
\begin{align*}
\pi_{1} T_{n}^{r}\binom{x}{y} & =\frac{2 x}{n^{2} a} \epsilon^{2}-\frac{4 \sqrt{V_{r}} y}{n^{3} a^{2}} \epsilon^{3}+O\left(\epsilon^{4}\right)  \tag{2.18}\\
\pi_{2} T_{n}^{r}\binom{x}{y} & =\frac{2 y}{n^{2} a} \epsilon^{2}+\frac{4 \sqrt{V_{r}} x}{n^{3} a^{2}} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{2.19}\\
\pi_{1} T_{n}^{l}\binom{x}{y} & =\frac{-2 x}{n^{2} a} \epsilon^{2}-\frac{4 \sqrt{V_{l} y}}{n^{3} a^{2}} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{2.20}\\
\pi_{2} T_{n}^{l}\binom{x}{y} & =\frac{-2 y}{n^{2} a} \epsilon^{2}+\frac{4 \sqrt{V_{l} x}}{n^{3} a^{2}} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{2.21}\\
S_{n}^{r}(x) & =\frac{2 x}{n^{2} a} \epsilon^{2}-\frac{4 \sqrt{-V_{r} x}}{n^{3} a^{2}} \epsilon^{3}+O\left(\epsilon^{4}\right)  \tag{2.22}\\
S_{n}^{l}(x) & =\frac{-2 x}{n^{2} a} \epsilon^{2}+\frac{4 \sqrt{-V_{l} x}}{n^{3} a^{2}} \epsilon^{3}+O\left(\epsilon^{4}\right) . \tag{2.23}
\end{align*}
$$

Proof. Formulas (2.18)-(2.23) are obtained by expanding the operators $T_{n}^{r}, T_{n}^{l}, S_{n}^{r}$, and $S_{n}^{l}$ in powers of $\epsilon$ using geometric series.

Corollary 1. If $x$ and $y$ are $O\left(\epsilon^{\alpha}\right)$, then

$$
\begin{align*}
p_{n}(T)\binom{x}{y} & =(-1)^{n+1} T_{n}\binom{x}{y}+\binom{O\left(\epsilon^{\alpha+4}\right)}{O\left(\epsilon^{\alpha+4}\right)},  \tag{2.24}\\
p_{n}(S)(x) & =(-1)^{n+1} S_{n}(x)+O\left(\epsilon^{\alpha+4}\right) \tag{2.25}
\end{align*}
$$

where superscripts $r$ and l have been suppressed.
Proof. The only degree one term in the polynomial $p_{n}(T)$ is $(-1)^{n+1} T_{n}$. The same is true for $p_{n}(S)$.

The derivations of the asymptotic expansions listed in Theorem 1 are presented next.

### 2.2 Asymptotic approximation in region I

We seek a solution of the form

$$
R^{\epsilon}(\xi ; \epsilon)= \begin{cases}A(\xi ; \epsilon) \exp \left(\sqrt{-V_{l}} \xi\right) & \xi<0  \tag{2.26}\\ B(\xi ; \epsilon) \exp \left(-\sqrt{-V_{r}} \xi\right) & \xi>0\end{cases}
$$

Substituting (2.26) into equation (2.4) gives the system

$$
\begin{align*}
& A^{\prime \prime}+2 \sqrt{-V_{l}} A^{\prime}+\left(V^{\epsilon}-V_{l}\right) A=0  \tag{2.27}\\
& B^{\prime \prime}-2 \sqrt{-V_{r}} B^{\prime}+\left(V^{\epsilon}-V_{r}\right) B=0 \tag{2.28}
\end{align*}
$$

These equations for $A(\xi ; \epsilon)$ and $B(\xi ; \epsilon)$ are at least as hard to solve as equation (2.4) for $R^{\epsilon}(\xi ; \epsilon)$. We therefore introduce the following expansions:

$$
\begin{align*}
& A(\xi ; \epsilon)=A_{0}(\epsilon)+\sum_{1}^{\infty} A_{n}(\epsilon) \exp \left(\frac{n a \xi}{\epsilon}\right),  \tag{2.29}\\
& B(\xi ; \epsilon)=B_{0}(\epsilon)+\sum_{1}^{\infty} B_{n}(\epsilon) \exp \left(\frac{-n a \xi}{\epsilon}\right), \tag{2.30}
\end{align*}
$$

where the coefficients have expansions in powers of $\epsilon$ :

$$
\begin{align*}
& A_{n}(\epsilon)=\sum_{j=0}^{\infty} A_{n j} \epsilon^{j} \quad n \geq 0,  \tag{2.31}\\
& B_{n}(\epsilon)=\sum_{j=0}^{\infty} B_{n j} \epsilon^{j} \quad n \geq 0 . \tag{2.32}
\end{align*}
$$

Several points are worth emphasizing:

1. The terms $A_{0}, B_{0}$ are the outer contributions, important away from $\xi=0$. They can be represented as algebraic expansions in $\epsilon$ as in (2.31), (2.32).
2. The terms $A_{n}, B_{n}, n \geq 1$, also have expansions in powers of $\epsilon$, but the expressions $\sum_{1}^{\infty} A_{n}(\epsilon) \exp \left(\frac{n a \xi}{\epsilon}\right)$ and $\sum_{1}^{\infty} B_{n}(\epsilon) \exp \left(\frac{-n a \xi}{\epsilon}\right)$ have summands which are transcendentally small for $\xi \neq 0$ as $\epsilon \rightarrow 0$. These terms constitute the inner contributions, important near $\xi=0$.
3. The continuity of the solution at $\xi=0$ involves all of the terms $A_{n}, B_{n}, n \geq$ 0 . This is because the inner terms and outer terms are of the same order of magnitude at $\boldsymbol{\xi}=0$. The expansions listed in Theorem 1 are constructed so as to satisfy the continuity conditions at $\xi=0$ with all of the transcendentally small terms included.

Equations (2.27) and (2.28) give algebraic equations for the coefficients $A_{n}(\epsilon)$ and $B_{n}(\epsilon)$. These algebraic equations are easily solved. The solutions are

$$
\begin{equation*}
A_{n}=p_{n}\left(S^{l}\right) A_{0} \quad n>0 \tag{2.33}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{n}=p_{n}\left(S^{\top}\right) B_{0} \quad n>0, \tag{2.34}
\end{equation*}
$$

where $p_{n}\left(S^{l}\right)$ and $p_{n}\left(S^{\top}\right)$ are defined in (2.16) and (2.17). The coefficients $A_{0}$ and $B_{0}$ will be determined by the matching condition

$$
\begin{equation*}
R^{\epsilon}\left(0^{-} ; \epsilon\right)=R^{\epsilon}\left(0^{+} ; \epsilon\right)=\alpha \tag{2.35}
\end{equation*}
$$

This matching condition is equivalent to the two conditions

1. $A(0 ; \epsilon)=\alpha$.
2. $B(0 ; \epsilon)=\alpha$.

By (2.29), condition 1 gives the equation

$$
\begin{equation*}
A_{0}(\epsilon)+\sum_{i}^{\infty} A_{n}(\epsilon)=\alpha \tag{2.36}
\end{equation*}
$$

By (2.33), this is

$$
\begin{equation*}
A_{0}+\sum_{1}^{\infty}\left(p_{n}\left(S^{l}\right) A_{0}\right)=\alpha \tag{2.37}
\end{equation*}
$$

With $A_{0}(\epsilon)=\sum_{0}^{\infty} A_{0 n} \epsilon^{n}$, (2.37) becomes

$$
\begin{equation*}
\sum_{0}^{\infty} A_{0 n} \epsilon^{n}+\sum_{1}^{\infty}\left(p_{n}\left(S^{l}\right)\left[\sum_{0}^{\infty} A_{0 n} \epsilon^{n}\right]\right)=\alpha \tag{2.38}
\end{equation*}
$$

Using Lemma 1 and Corollary 1, we collect powers of $\epsilon$ on both sides of the equation. The first three equations obtained are:

$$
\begin{aligned}
& O(1): A_{00}=\alpha, \\
& O(\epsilon): A_{01}=0, \\
& O\left(\epsilon^{2}\right): A_{02}+\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{2} a}\left(-2 A_{00}\right)=0 .
\end{aligned}
$$

The result is

$$
\begin{equation*}
A_{0}(\epsilon)=\alpha\left(1+\frac{\pi^{2}}{6 a} \epsilon^{2}+O\left(\epsilon^{3}\right)\right) \tag{2.39}
\end{equation*}
$$

Therefore, $A(\xi ; \epsilon)=A_{0}(\epsilon)+$ t.s.t. as claimed in the first paragraph of Theorem 1. Matching condition 2 gives the equation

$$
\begin{equation*}
B_{0}+\sum_{1}^{\infty}\left(p_{n}\left(S^{\top}\right) B_{0}\right)=\alpha \tag{2.40}
\end{equation*}
$$

With $B_{0}=\sum_{0}^{\infty} B_{0 n} \epsilon^{n}$, we have the equations

$$
\begin{aligned}
& O(1): B_{00}=\alpha \\
& O(\epsilon): B_{01}=0 \\
& O\left(\epsilon^{2}\right): B_{02}+\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{2} a}\left(2 B_{00}\right)=0,
\end{aligned}
$$

which results in

$$
\begin{equation*}
B_{0}(\epsilon)=\alpha\left(1-\frac{\pi^{2}}{6 a} \epsilon^{2}++O\left(\epsilon^{3}\right)\right) \tag{2.41}
\end{equation*}
$$

as in Theorem 1. This concludes the region I analysis.

### 2.3 Asymptotic approximation in region II

In region II we seek a solution in the form

$$
R^{\epsilon}(\xi)= \begin{cases}A(\xi ; \epsilon) \exp \left(\sqrt{-V_{l}} \xi\right) & \xi<0  \tag{2.42}\\ B(\xi ; \epsilon) \sin \left(\sqrt{V_{r}} \xi\right)+C(\xi ; \epsilon) \cos \left(\sqrt{V_{r}} \xi\right) & \xi>0\end{cases}
$$

The equations for $A(\xi ; \epsilon), B(\xi ; \epsilon)$, and $C(\xi ; \epsilon)$ are

$$
\begin{equation*}
A^{\prime \prime}+2 \sqrt{-V_{l}} A^{\prime}+\left(V^{\epsilon}-V_{l}\right) A=0, \tag{2.43}
\end{equation*}
$$

and the coupled system

$$
\begin{aligned}
& B^{\prime \prime}-2 \sqrt{-V_{r}} C^{\prime}+\left(V^{\epsilon}-V_{r}\right) B=0 \\
& C^{\prime \prime}+2 \sqrt{-V_{r}} B^{\prime}+\left(V^{\epsilon}-V_{r}\right) C=0
\end{aligned}
$$

With $C(\xi ; \epsilon)=\sum_{0}^{\infty} C_{n}(\epsilon) \exp (-n a \xi / \epsilon)$, and $A, B$ as before, we find

$$
\begin{equation*}
A_{n}=p_{n}\left(S^{l}\right) A_{0} \quad n>0, \tag{2.44}
\end{equation*}
$$

and

$$
\begin{equation*}
\binom{B_{n}}{C_{n}}=p_{n}\left(T^{*}\right)\binom{B_{0}}{C_{0}} \quad n>0 . \tag{2.45}
\end{equation*}
$$

The matching conditions in region II are

1. $A(0 ; \epsilon)=\alpha$.
2. $C(0 ; \epsilon)=\alpha$.
3. $A^{\prime}(0 ; \epsilon)+\sqrt{-V_{l}} A(0 ; \epsilon)=\sqrt{V_{r}} B(0 ; \epsilon)+C^{\prime}(0 ; \epsilon)$.

Condition 3 is the requirement that $\frac{d R^{s}}{d \xi}\left(0^{-}\right)=\frac{d R^{*}}{d \xi}\left(0^{+}\right)$. As in the region I analysis, the result of condition 1 is

$$
\begin{equation*}
A_{0}(\epsilon)=\alpha\left(1+\frac{\pi^{2}}{6 a} \epsilon^{2}+O\left(\epsilon^{3}\right)\right) \tag{2.46}
\end{equation*}
$$

Condition 2 gives the equation

$$
\begin{equation*}
C_{0}+\sum_{i}^{\infty} \pi_{2} p_{n}\left(T^{r}\right)\binom{B_{0}}{C_{0}}=\alpha \tag{2.47}
\end{equation*}
$$

which implies

$$
\begin{aligned}
& O(1): C_{00}=\alpha, \\
& O(\epsilon): C_{01}=0, \\
& O\left(\epsilon^{2}\right): C_{02}+\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{2} a}\left(2 C_{00}\right)=0 .
\end{aligned}
$$

The result is

$$
\begin{equation*}
C_{0}(\epsilon)=\alpha\left(1-\frac{\pi^{2}}{6 a} \epsilon^{2}+O\left(\epsilon^{3}\right)\right) . \tag{2.48}
\end{equation*}
$$

Condition 3 gives the equation

$$
\begin{align*}
\sum_{1}^{\infty}\left(\frac{n a}{\epsilon}\right) p_{n}\left(S^{l}\right) A_{0}+\sqrt{-V_{l}} \alpha & =\sqrt{V_{T}}\left\{B_{0}+\sum_{1}^{\infty} \pi_{1} p_{n}\left(T^{r}\right)\binom{B_{0}}{C_{0}}\right\}  \tag{2.49}\\
& +\sum_{1}^{\infty}\left(\frac{-n a}{\epsilon}\right) \pi_{2} p_{n}\left(T^{r}\right)\binom{B_{0}}{C_{0}}
\end{align*}
$$

Setting powers of $\epsilon$ on both sides of (2.49) equal to zero gives

$$
\begin{aligned}
& O(1): \sqrt{-V_{l}}=\sqrt{V_{r}} B_{00}, \\
& O(\epsilon): \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}\left(-2 A_{00}\right)=\sqrt{V_{r}} B_{01}+\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}\left(2 C_{00}\right), \\
& O\left(\epsilon^{2}\right): \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{2}}\left(4 \sqrt{-V_{l}} A_{00}\right)=\sqrt{V_{r}} B_{02}+\sqrt{V_{r}} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{2} a}\left(2 B_{00}\right) \\
& -\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{2} a}\left(4 \sqrt{V_{r}} B_{00}\right) .
\end{aligned}
$$

The result is

$$
\begin{equation*}
B_{0}(\epsilon)=\alpha \sqrt{\frac{-V_{i}}{V_{T}}}\left(1+\frac{\pi^{2}}{2 a} \epsilon^{2}+O\left(\epsilon^{3}\right)\right) . \tag{2.50}
\end{equation*}
$$

This concludes the region II analysis.

### 2.4 Asymptotic approximation in region III

The expansion in this region takes the form

$$
R^{\epsilon}(\xi)= \begin{cases}A(\xi ; \epsilon) \sin \left(\sqrt{V_{l}} \xi\right)+B(\xi ; \epsilon) \cos \left(\sqrt{V_{l}} \xi\right) & \xi<0  \tag{2.51}\\ C(\xi ; \epsilon) \sin \left(\sqrt{V_{r}} \xi\right)+D(\xi ; \epsilon) \cos \left(\sqrt{V_{r}} \xi\right) . & \xi>0 .\end{cases}
$$

The equations for $A, B, C$, and $D$ are

$$
\begin{aligned}
& A^{\prime \prime}-2 \sqrt{V_{l}} B^{\prime}+\left(V^{\epsilon}-V_{l}\right) A=0 \\
& B^{\prime \prime}+2 \sqrt{V_{l}} A^{\prime}+\left(V^{\epsilon}-V_{l}\right) B=0,
\end{aligned}
$$

and

$$
\begin{aligned}
& C^{\prime \prime}-2 \sqrt{V_{r}} D^{\prime}+\left(V^{\epsilon}-V_{r}\right) C=0 \\
& D^{\prime \prime}+2 \sqrt{-V_{r}} C^{\prime}+\left(V^{\epsilon}-V_{r}\right) D=0
\end{aligned}
$$

The solutions of the resulting algebraic equations are

$$
\begin{align*}
& \binom{A_{n}}{B_{n}}=p_{n}\left(T^{d}\right)\binom{A_{0}}{B_{0}}, \quad n>0  \tag{2.52}\\
& \binom{C_{n}}{D_{n}}=p_{n}\left(T^{*}\right)\binom{C_{0}}{D_{0}}, \quad n>0 . \tag{2.53}
\end{align*}
$$

The matching conditions are

1. $B(0 ; \epsilon)=\alpha$.
2. $D(0 ; \epsilon)=\alpha$.
3. $\sqrt{ } \nabla_{l} A(0 ; \epsilon)+B^{\prime}(0 ; \epsilon)=\beta$.
4. $\sqrt{\nabla_{\Gamma}} C(0 ; \epsilon)+D^{\prime}(0 ; \epsilon)=\beta$.

Conditions 3 and 4 are equivalent to $\frac{d R^{e}}{d \xi}\left(0^{-}\right)=\frac{d R^{e}}{d \xi}\left(0^{+}\right)=\beta$. The remaining analysis is essentially the same as the region I and II analysis and so is omitted. This concludes the derivations of the expansions in Theorem 1.

## 3 Dynamics prior to shock onset

We turn again to system (1.0) for $u(x, t), E(x, t)$, but now with initial data (Figure 6) for $u$ :

$$
u(x, 0)= \begin{cases}u_{l} & x<-\frac{h}{2}  \tag{3.1}\\ u_{l}-\frac{2 a}{h}\left(x+\frac{h}{2}\right) & x \in\left(-\frac{h}{2}, \frac{h}{2}\right) \\ u_{r} & x>\frac{h}{2}\end{cases}
$$



Figure 6: Initial data (3.1).
where $a=\frac{1}{2}\left(u_{l}-u_{r}\right)>0$, and $h$ is a fixed but arbitrary positive constant. It is well known [1, 2] that the initial data (3.1) evolves according to the conservation law (1.0a) into a shock steady in the $\xi=x-c t$ variable, $c=\frac{1}{2}\left(u_{l}+u_{r}\right)>0$. The shock formation occurs at time $t^{*}=h / 2 a$. In this section, a family of piecewise smooth solutions of (1.0), (3.1) is described. One feature of these solutions is that $\theta(\xi, t)$ is piecewise linear in $\xi$. Though $\theta(\xi, t)$ does evolve in time, its spatial structure remains piecewise linear. On a certain interval described later, the slope of $\theta(\xi, t)$ becomes infinite as $u(\xi, t)$ forms a shock. These solutions thus show how shock formation in $u$ can drive a finite time blow-up in $\theta_{\xi}$. Since $\theta_{\xi}$ can be interpreted as a generalized wavenumber for the modulated wave $R \exp (i \theta)$, the result shows that shock formation of $u$ provides a mechanism by which small scales (high wavenumbers) are generated in the dispersive system. Moreover, we show that the rate at which the wave number becomes infinite is $O\left(\log \left(t^{*}-t\right)\right)$ as $t \rightarrow t^{*}$ while the shock forms at the rate $O\left(1 /\left(t^{*}-t\right)\right)$ as $t \rightarrow t^{*}$.

### 3.1 The general procedure: variation of parameters

For ease of notation, let 'gap' denote the time dependent interval ( $-h / 2+a t, h / 2-a t$ ), which is centered about 0 and collapses to 0 as $t \rightarrow h / 2 a$. By $\xi<$ gap, we mean points
to the left of this interval, etc. The solution of (1.0a), (3.1) can then be written as

$$
\left.\begin{array}{l}
u(\xi, t)=\left\{\begin{array}{ll}
u_{l} & \xi<\text { gap } \\
c-\frac{\xi}{t^{*}-t} & \xi \in \text { gap } \\
u_{r} & \xi>\text { gap }
\end{array} \quad \text { for } t<t^{*},\right.
\end{array}\right\} \begin{aligned}
& u(\xi, t)= \begin{cases}u_{l} & \xi<0 \\
u_{r} & \xi>0\end{cases}
\end{aligned}
$$

where $t^{*}=h / 2 a$. The evolution of $u(\xi, t)$ given by (3.2) is particularly simple: as $t \rightarrow t^{*}$, the slanted line segment in Figure 6 steepens as the interval ( $-h / 2+a t, h / 2-a t$ ) collapses to the origin. Then at time $t^{*}, u(\xi, t)$ becomes the steady-state step function (1.2) considered in section 1. Motivated by equation (1.5), we seek $\theta(\xi, t)$ in the form

$$
\begin{equation*}
\theta(\xi, t)=\omega(t) \xi+\chi(t) \tag{3.3}
\end{equation*}
$$

This is simply a variation of parameters applied to (1.5). We break up $\theta(\xi, t)$ into three parts corresponding to the intervals on which $u(\xi, t)$ is linear:

$$
\theta(\xi, t)= \begin{cases}\omega_{1}(t) \xi+\chi_{1}(t) & \xi<\text { gap }  \tag{3.4}\\ \omega_{2}(t) \xi+\chi_{2}(t) & \xi \in \text { gap } \\ \omega_{3}(t) \xi+\chi_{3}(t) & \xi>\text { gap }\end{cases}
$$

and use (1.0b) to derive equations for $\omega_{i}$ and $\chi_{i}, i=1,2,3$. We shall see that $\left|\omega_{2}(t)\right| \rightarrow$ $\infty$ as $t \rightarrow t^{*}$. For the amplitude $R$, let

$$
R(\xi, t)= \begin{cases}R_{1}(\xi, t) & \xi<\text { gap }  \tag{3.5}\\ R_{2}(\xi, t) & \xi \in \text { gap } \\ R_{3}(\xi, t) & \xi>\text { gap. }\end{cases}
$$

### 3.2 Derivation of solutions

Equation (1.0b) in phase-amplitude form is

$$
\begin{align*}
& \frac{\partial R}{\partial t}+\left(2 \frac{\partial \theta}{\partial \xi}-c\right) \frac{\partial R}{\partial \xi}+\frac{\partial^{2} \theta}{\partial \xi^{2}} R=0  \tag{3.6a}\\
& \frac{\partial^{2} R}{\partial \xi^{2}}+\left(c \frac{\partial \theta}{\partial \xi}-\left(\frac{\partial \theta}{\partial \xi}\right)^{2}-\frac{\partial \theta}{\partial t}-u\right) R=0 . \tag{3.6b}
\end{align*}
$$

Substituting $\theta=\omega_{i}(t) \xi+\chi_{i}(t)$ and $R=R_{i}$ into (3.6) gives

$$
\begin{align*}
& \frac{\partial R_{i}}{\partial t}=\left(c-2 \omega_{i}\right) \frac{\partial R_{i}}{\partial \xi}  \tag{3.7a}\\
& \frac{\partial^{2} R_{i}}{\partial \xi^{2}}+\left(c \omega_{i}-\omega_{i}^{2}-\dot{\omega}_{i} \xi-\dot{\chi}_{i}-u\right) R_{i}=0 \tag{3.7b}
\end{align*}
$$

where $\cdot=d / d t$. Equation (3.7a) is a first order wave equation for $R_{i}$ with variable wavespeed. As long as $R_{i}$ depends only on $\xi+\eta_{i}$, with $\eta_{i}=\int_{0}^{t}\left(c-2 \omega_{i}(s)\right) d s$, equation (3.7a) will be satisfied. The potential $\omega_{i}-\omega_{i}^{2}-\dot{\omega}_{i} \xi-\dot{\chi}_{i}-u$ in (3.7b) must then be a function only of the variable $\xi+\eta_{i}$. We state a simple lemma:

Lemma 2. If $f, g$ and $h$ are differentiable functions of a single variable and if the relation

$$
\begin{equation*}
h(t) \xi+g(t)=f\left(\xi+\eta_{i}\right) \tag{3.8}
\end{equation*}
$$

holds for $t \geq 0$ and all $\xi$, then $h(t)=\lambda, \lambda$ constant, and $g(t)=\lambda \eta_{i}+\mu, \mu$ constant.

Proof. Differentiating both sides of (3.8) with respect to $\xi$ gives

$$
\begin{equation*}
h(t)=f^{\prime}\left(\xi+\eta_{i}\right), \tag{3.9}
\end{equation*}
$$

where prime is differentiation with respect to $\xi+\eta_{i}$. Since the left side of (3.9) is independent of $\xi$, both sides of (3.9) are equal to a constant, say $\lambda$. Hence $f\left(\xi+\eta_{i}\right)=$ $\lambda\left(\xi+\eta_{i}\right)+\mu$ for some constant $\mu$, and the lemma is proved.

We use Lemma 2 to derive equations for $\omega_{i}$ and $\chi_{i}, i=1,2,3$, so that the potential

$$
\begin{equation*}
\left(\omega_{i}-\omega_{i}^{2}-\dot{\omega}_{i} \xi-\dot{\chi}_{i}-u\right) \tag{3.10}
\end{equation*}
$$

in (3.7b) depends as generally as possible on the single variable $\xi+\eta_{i}$. Each region is treated separately.

For $\xi<$ gap and $t<t^{*}, u(\xi, t)$ is the constant $u_{l}$. Applying the lemma with $h(t)=-\dot{\omega}_{1}$, and $g(t)=\left(\omega_{1}-\omega_{1}^{2}-\dot{\chi}_{1}-u_{l}\right)$ gives

$$
\begin{align*}
& \dot{\omega}_{1}=\lambda_{1}  \tag{3.11}\\
& \dot{\chi}_{1}=c \omega_{1}-\omega_{1}^{2}-u_{l}+\lambda_{1} \eta_{1}-\mu_{1}, \tag{3.12}
\end{align*}
$$

where $\lambda_{1}$ and $\mu_{1}$ are the constants from Lemma 2. Integrating (3.11) and (3.12) leads to

$$
\begin{align*}
& \omega_{1}=\lambda_{1} t+\beta_{1}  \tag{3.13}\\
& \chi_{1}=\frac{-2 \lambda_{1}}{3} t^{3}+\lambda_{1}\left(c-2 \beta_{1}\right) t^{2}+\left(c \beta_{1}-\beta_{1}^{2}-u_{1}-\mu_{1}\right) t+\gamma_{1} \tag{3.14}
\end{align*}
$$

where $\beta_{1}$ and $\gamma_{1}$ are constants of integration. Similarly, for $\xi>$ gap and $t<t^{*}$, we have

$$
\begin{align*}
& \omega_{3}=\lambda_{3} t+\beta_{3}  \tag{3.15}\\
& \chi_{3}=\frac{-2 \lambda_{3}}{3} t^{3}+\lambda_{3}\left(c-2 \beta_{3}\right) t^{2}+\left(c \beta_{3}-\beta_{3}^{2}-u_{r}-\mu_{3}\right) t+\gamma_{3}, \tag{3.16}
\end{align*}
$$

where $\lambda_{3}, \mu_{3}, \beta_{3}$, and $\gamma_{3}$ are constants. In the gap region, $u(\xi, t)=c-\xi /\left(t^{*}-t\right)$. Applying the lemma with $h(t)=-\dot{\omega}_{2}+\left(t^{*}-t\right)^{-1}$ and $g(t)=\left(\omega_{2}-\omega_{2}^{2}-\dot{\chi}_{2}-c\right)$ gives

$$
\begin{align*}
& \dot{\omega}_{2}=\lambda_{2}+\left(t^{*}-t\right)^{-1}  \tag{3.17}\\
& \dot{\chi}_{2}=c \omega_{2}-\omega_{2}^{2}-c+\lambda_{2} \eta_{2}-\mu_{2}, \tag{3.18}
\end{align*}
$$

where $\lambda_{2}$ and $\mu_{2}$ are constants. This gives

$$
\begin{gather*}
\omega_{2}=\lambda_{2} t+\beta_{2}-\log \left(t^{*}-t\right)  \tag{3.19}\\
\chi_{2}=\int^{t} F  \tag{3.20}\\
F=\left(c \beta_{2}-\beta_{2}^{2}-c-\mu_{2}\right)+2 \lambda_{2}\left(c-2 \beta_{2}+1\right) t-2 \lambda_{2}^{2} t^{2}  \tag{3.21}\\
+\left(2 \beta_{2}-c-2 \lambda_{2} t^{*}\right) \log \left(t^{*}-t\right)+4 \lambda_{2} t \log \left(t^{*}-t\right)-\log ^{2}\left(t^{*}-t\right),
\end{gather*}
$$

where $\beta_{2}$ is constant. Note that (3.19) implies $\left|\omega_{2}(t)\right| \rightarrow \infty$ logarithmically as $t \rightarrow t^{*}$. Also note (3.17), (3.18) show that $\dot{\omega}_{2}$ and $\dot{\chi}_{2}$ become infinite as $t \rightarrow t^{*}$.

We can now substitute $\omega_{i}$ and $\chi_{i}, i=1,2,3$ into (3.7b) to derive the governing equation for the amplitudes $R_{i}$. The result is

$$
\begin{equation*}
R_{i}^{\prime \prime}+\left\{-\lambda_{i}\left(\xi+\eta_{i}\right)+\mu_{i}\right\} R_{i}=0, \quad i=1,2,3, \tag{3.22}
\end{equation*}
$$

where prime denotes differentiation with respect to $\xi+\eta_{i}$. If $\lambda_{i} \neq 0$, then (3.22) is Airy's equation and has the solution

$$
\begin{equation*}
R_{i}\left(\xi+\eta_{i}\right)=c_{0} A i\left\{\lambda_{i}^{-2 / 3}\left(\lambda_{i}\left(\xi+\eta_{i}\right)-\mu_{i}\right)\right\}+c_{1} B i\left\{\lambda_{i}^{-2 / 3}\left(\lambda_{i}\left(\xi+\eta_{i}\right)-\mu_{i}\right)\right\} \tag{3.23}
\end{equation*}
$$

where $c_{0}, c_{1}$ are constants.
Although we do not have continuity in $\xi$ for the solutions $R(\xi, t)$ and $\theta(\xi, t)$, we can enforce temporal continuity through break time $t^{*}=h / 2 a$ by defining

$$
\begin{align*}
R(\xi, t) & = \begin{cases}R_{1}\left(\xi+\eta_{1}(t)\right) & \xi<0 \\
R_{3}\left(\xi+\eta_{3}(t)\right) & \xi>0\end{cases}
\end{align*} \quad t \geq t^{*}, ~ 子 \begin{array}{ll} 
 \tag{3.24}\\
\theta(\xi, t) & = \begin{cases}\omega_{1}(t) \xi+\chi_{1}(t) & \xi<0 \\
\omega_{3}(t) \xi+\chi_{3}(t) & \xi>0\end{cases} \tag{3.25}
\end{array}
$$

We have thus described a family of spatially piecewise smooth solutions of (1.0), (3.1) which are defined for $t \geq 0$. For these solutions, the phase $\theta(\xi, t)$ becomes infinitely steep on the collapsing interval $(-h / 2+a t, h / 2+a t)$ as $u(\xi, t)$ shocks.

### 3.3 Transients of the Riemann problem

As $t \rightarrow t^{*}=h / 2 a$, the initial data (3.1) for $u(\xi, t)$ evolves to (1.1), the initial data of the Riemann problem studied in section 1 . In this sense, system (1.0), (3.1) extends the Riemann problem back in time to a state prior to shock formation. The solutions of (1.0), (3.1) derived in the previous section undergo phase gradient blow-up in the
interval ( $-h / 2 a+a t, h / 2 a-a t$ ), and then have evolutions given by (3.2b), (3.24), (3.25). The following question then arises: Do any solutions of (1.0), (3.1) described in the previous section coincide, for $\pm \geq t^{*}$, with the solutions (1.5), (1.7) of the Riemann problem? The answer is yes. To identify which solutions evolve to the steady states of section 1 , it suffices to find conditions under which (3.24) and (3.25), the governing equations for $R$ and $\theta$ beyond $t^{*}$, coincide with (1.5) and (1.6), the equations governing the steady-state solutions. Obviously, if solutions are going to coincide with steadystates beyond $t^{*}$, they can not depend explicitly on $t$ beyond $t^{*}$. This condition is, in fact, sufficient: we will show that as long as $R$ and $\theta$, given by (3.24), (3.25), have no explicit $t$ dependence beyond time $t^{*}$, they correspond to solutions (1.5), (1.7) of section 1. These are the transients of solutions (1.5), (1.7) (see Figures 7 and 8). The solutions which have explicit time dependence beyond $t^{*}$ can still be considered as solutions (for $t \geq t^{*}$ ) of the Riemann problem, but they are not transients of the steady-states of section 1 (see Figure 9).

The solutions derived in the previous section are determined by the parameters $\lambda_{i}$, $\beta_{i}, \mu_{i}$, and $\gamma_{i}, i=1,2,3$. By equation (3.24), $R(\xi, t)$ will be a function of $\xi$ alone for $t \geq t^{*}$ precisely when $d \eta_{i} / d t=0, i=1,3$. Since $\eta_{i}(t)=\int_{0}^{t}\left(c-2 \omega_{i}(s)\right) d s$, this is just the condition that $\omega_{i}=c / 2, i=1,3$. By (3.13) and (3.15), this is equivalent to $\lambda_{i}=0$, $\beta_{i}=c / 2, i=1,3$. Thus the steadiness of the amplitude for $t \geq t^{*}$ is equivalent to the phase having the form (for $t \geq t^{*}$ )

$$
\theta(\xi, t)= \begin{cases}c \xi / 2+\left(c^{2} / 4-u_{l}-\mu_{1}\right) t+\gamma_{1} & \xi<0  \tag{3.26}\\ c \xi / 2+\left(c^{2} / 4-u_{r}-\mu_{3}\right) t+\gamma_{3} & \xi>0\end{cases}
$$

and the amplitude satisfying

$$
\begin{array}{ll}
R_{1}^{\prime \prime}+\mu_{1} R_{1}=0 & \xi<0 \\
R_{3}^{\prime \prime}+\mu_{3} R_{3}=0 & \xi>0, \tag{3.28}
\end{array}
$$

where ' $=d / d \xi$. If we additionally require the phase to be a function of $\xi$ alone for
$t \geq t^{*}$, we must set

$$
\begin{aligned}
& \mu_{l}=c^{2} / 4-u_{l}=V_{l}(\text { defined after (1.6)) } \\
& \mu_{3}=c^{2} / 4-u_{r}=V_{r}
\end{aligned}
$$

in (3.26). Then for $t \geq t^{*}$,

$$
\theta(\xi, t)= \begin{cases}c \xi / 2+\gamma_{1} & \xi<0  \tag{3.29}\\ c \xi / 2+\gamma_{3} & \xi>0\end{cases}
$$

and

$$
\begin{align*}
& R_{1}^{\prime \prime}+V_{l} R_{1}=0  \tag{3.30}\\
& R_{3}^{\prime \prime}+V_{\Gamma} R_{3}=0 . \tag{3.31}
\end{align*}
$$

These are exactly equations (1.5), (1.6) governing the steady-states in section 1 . Thus transients correspond to parameter values $\lambda_{i}=0, \beta_{i}=c / 2,(i=1,3)$, and $\mu_{1}=V_{l_{1}}$ $\mu_{3}=V_{r}$. We remark that no conditions on parameters with subseript 2 are given. These parameters determine the phase and amplitude in the gap region which vanishes at time $t^{*}$ and so play no role in the evolution beyond $t^{*}$.

### 3.4 Some phase-amplitude plots through break time

Figures 7, 8, and 9 show amplitude and phase evolutions which become steady state, phase-kinked steady state, and non-steady state, respectively, at break time. We have selected $u_{l}=6$ and $u_{r}=2$, which corresponds to shock values in region II (see Figure 1 , page 9). The length parameter $h$ as defined in equation (3.1) is 2 which implies that break time is $t^{*}=1 / 2$. Figure 8 shows only the phase evolution since the amplitude in this example is the same as for Figure 7. The amplitude in Figure 7 is given by

$$
\begin{align*}
& t \leq 1 / 2: R(\xi, t)=\left\{\begin{array}{lc}
\exp (\sqrt{2} \xi) & \xi \leq-1+2 t \\
1 & -1+2 t<\xi<1-2 t \\
\sin (\sqrt{2} \xi)+\cos (\sqrt{2} \xi) & 1-2 t \leq \xi
\end{array}\right.  \tag{3.32}\\
& t>1 / 2: R(\xi, t)= \begin{cases}\exp (\sqrt{2} \xi) & \xi<0 \\
\sin (\sqrt{2} \xi)+\cos (\sqrt{2} \xi) & \xi>0 .\end{cases}
\end{align*}
$$

The phase in Figure 7 is given by

$$
\begin{align*}
& t \leq 1 / 2: \theta(\xi, t)=\left\{\begin{array}{lc}
2 \xi & \xi \leq-1+2 t \\
(2-\log 2-\log (1 / 2-t)) \xi & -1+2 t<\xi<1-2 t \\
2 \xi & 1-2 t \leq \xi
\end{array}\right.  \tag{3.33}\\
& t>1 / 2: \theta(\xi, t)= \begin{cases}2 \xi & \xi<0 \\
2 \xi & \xi>0 .\end{cases}
\end{align*}
$$

The actual phase would include the term $\chi_{2}(t)$ in the interval $-1+2 t<\xi<1-2 t$ (see equations (3.3) and (3.20)). With the parameter values as specified, this term is

$$
\begin{align*}
\chi_{2}(t) & =(0.5-t)\left(2-2 \log 2+(\log 2)^{2}\right)+2(0.5-t)(\log 2-1) \log (0.5-t)  \tag{3.34}\\
& +(0.5-t) \log (0.5-t)^{2} .
\end{align*}
$$

For ease of graphing, we have set $\chi_{2}(t)=0$ in each figure. As a result, the phase always passes through the origin.

The phase in Figure 8 goes to a kinked steady state. It is

$$
\begin{align*}
& t \leq 1 / 2: \theta(\xi, t)=\left\{\begin{array}{lc}
2 \xi-5 & \xi \leq-1+2 t \\
(7-\log 2-\log (1 / 2-t)) \xi & -1+2 t<\xi<1-2 t \\
2 \xi+5 & 1-2 t \leq \xi
\end{array}\right.  \tag{3.35}\\
& t>1 / 2: \theta(\xi, t)= \begin{cases}2 \xi-5 & \xi<0 \\
2 \xi+5 & \xi>0\end{cases}
\end{align*}
$$

We have excluded

$$
\begin{align*}
\chi_{2}(t) & =(0.5-t)\left(37-12 \log 2+(\log 2)^{2}\right)+2(0.5-t)(\log 2-6) \log (0.5-t) \\
& +(0.5-t) \log (0.5-t)^{2} . \tag{3.36}
\end{align*}
$$

Figure 9 shows an example of phase and amplitude evolutions through break time $t^{*}=1 / 2$ which never reach steady states. The amplitude in Figure 9 is a function of the variable $\xi+\eta, \eta=-2 t$, and is given by

$$
\begin{align*}
& t \leq 1 / 2: R(\xi, t)=\left\{\begin{array}{lc}
\exp (\sqrt{2}(\xi-2 t)) & \xi \leq-1+2 t \\
1 & -1+2 t<\xi<1-2 t \\
\sin (2(\xi-2 t)) & 1-2 t \leq \xi
\end{array}\right.  \tag{3.37}\\
& t>1 / 2: R(\xi, t)= \begin{cases}\exp (\sqrt{2}(\xi-2 t)) & \xi<0 \\
\sin (2(\xi-2 t)) & \xi>0 .\end{cases}
\end{align*}
$$

The phase is

$$
\begin{align*}
& t \leq 1 / 2: \theta(\xi, t)=\left\{\begin{array}{lc}
3 \xi-t & \xi \leq-1+2 t \\
(3-\log 2-\log (1 / 2-t)) \xi & -1+2 t<\xi<1-2 t \\
3 \xi-3 t & 1-2 t \leq \xi,
\end{array}\right. \\
& t>1 / 2: \theta(\xi, t)= \begin{cases}3 \xi-t & \xi<0 \\
3 \xi-3 t & \xi>0\end{cases} \tag{3.38}
\end{align*}
$$

We have excluded

$$
\begin{align*}
\chi_{2}(t) & =(0.5-t)\left(5-4 \log 2+(\log 2)^{2}\right)+2(0.5-t)(\log 2-2) \log (0.5-t)  \tag{3.39}\\
& +(0.5-t) \log (0.5-t)^{2}
\end{align*}
$$



Figure 7: Amplitude evolution to steady state. (a) Amplitude (3.32) at $t=0$. (b) $t=0.3$. (c) $t \geq 0.5$.


Figure 7: Phase evolution to steady state. (d) Phase (3.33) at $t=0$. (e) $t=0.3$. (f) $t \geq 0.5$.


Figure 8: Phase evolution to kinked steady state. (a) Phase (3.35) at $t=0$. (b) $t=0.3$. (c) $t \geq 0.5$.


Figure 9: Amplitude evolution to non-steady state. (a) Amplitude (3.37) at $t=0$. (b) $t=0.5$. (c) $t=1$.5.


Figure 9: Phase evolution to non-steady state. (d) Phase (3.38) at $t=0$. (e) $t=0.5$. (f) $t=1.5$.

## Chapter 2

## The incompressible limit

The fully coupled equations are obtained by setting $\delta=1$ in (0.9a). They are

$$
\begin{align*}
& \epsilon u_{t}+u\left(u+|E|^{2}\right)_{x}=0  \tag{4.0a}\\
& i E_{t}+E_{x x}-u E=0, \tag{4.0b}
\end{align*}
$$

with initial data given by $u(x, 0)=U(x), E(x, 0)=E_{0}(x)$. In the incompressible limit ( $\epsilon \rightarrow 0$ ), two widely separated time scales are discernible: a fast time scale of the hyperbolic wave $u$, and a slow time scale of the dispersive wave envelope $E$. This is seen by setting $\tau=t / \epsilon$ in (4.0a), which absorbs the coefficient $\epsilon$. Section 4 lists some exact solutions of system (4.0) which give insight into the nature of the nonlinear coupling and the various phenomena the system supports. In section 5, the two solutions of the $\epsilon=0$ problem for (4.0) are presented: one in which $E$ satisfies the cubic nonlinear Schrödinger equation, and one in which $E$ satisfies the linear Schrödinger equation. This leads us to seek two distinct asymptotic approximations to (4.0). In sections 6 and 7, two multi-time-scale expansions for $u$ and $E$ are constructed, one about each of the $\epsilon=0$ solutions, and the equations governing the leading order dispersive wave terms are solved. The solutions are the well-known soliton solution of NLS and an analogous solution of the linear Schrödinger equation. In sections 8 and 9 , the equations for the leading order term in the expansions for $u$, what we call the effective shock equations, are analyzed in detail. These equations determine how $u$ is influenced by $E$, and also how the phase of $E$ is affected by the rapidly moving shock. We show that to leading order, if the shock strength is weak compared to the soliton amplitude, the shock can be completely blocked. Section 10 consists of a series of numerical experiments which exemplify the shock phenomena described in sections 8 and 9.

## 4 Exact solutions

Although it is not possible to solve (4.0) for arbitrary initial data, there are some special solutions which give insight into the phenomena the system supports.

Example 1: Decoupled solutions.
We start by looking at two decoupled cases: $u=0, E \neq 0$, and $u \neq 0, E=0$. In the absence of a shock wave, i.e. $u=0$, the system reduces to the linear Schrödinger equation for the dispersive wave envelope:

$$
\begin{equation*}
i E_{t}+E_{x x}=0, \tag{4.1}
\end{equation*}
$$

which is satisfied by a linear superposition of plane waves:

$$
\begin{equation*}
E=\int_{-\infty}^{\infty} a(\omega) \exp (i(\sqrt{\omega} x-\omega t)) d \omega . \tag{4.2}
\end{equation*}
$$

To satisfy the initial data, we require that $E_{0}(x)=\int a(\omega) \exp (i \sqrt{\omega} x) d \omega$. This decoupled solution also solves (4.0) when $\epsilon=0$. It is considered again in section 5. An analogous solution can be obtained by setting $u=c$, where $c$ is a constant. When the dispersive wave is absent, i.e. $E=0$, the system reduces to the conservation law

$$
\begin{equation*}
\epsilon u_{t}+u u_{x}=0 \tag{4.3}
\end{equation*}
$$

which we know shocks in finite time [1] as long as $U^{\prime}(x)<0$ at some point $x$.

## Example 2: Coupled standing wave.

The fully coupled problem supports a much richer set of nonlinear solutions. From the Zakharov system [25], we expect that our system should support a fully coupled standing wave. It is straightforward to verify that an exact solution of this kind is given by

$$
\begin{align*}
u & =G+B^{2}\left(1-2 \operatorname{sech}^{2}(B x)\right)  \tag{4.4}\\
E & =2 B \operatorname{sech}(B x) \exp (-i G t) . \tag{4.5}
\end{align*}
$$

Notice that since $u$ is independent of time, the parameter $\epsilon$ does not play a role in the solution, hence this solution is somewhat special and is not expected to arise as a result
of a general wave interaction. In fact, (4.4), (4.5) solve (4.0) in the case $\epsilon=0$; (4.4) is a solution of the form $u=$ constant $-|E|^{2}$.

Example 3: Finite time blow-up.
Motivated by the solutions to the weakly coupled system described in section 3.2 of Chapter 1, a natural question to ask is whether the full system can support solutions in which the wave $u$ shocks in finite time. A solution of (4.0) demonstrating such shock development is constructed in the following way. Specify initial data (Figure 6, page 29)

$$
u(x, 0)= \begin{cases}u_{l} & x<-\frac{h}{2}  \tag{4.6}\\ u_{l}-\frac{2 a}{h}\left(x+\frac{h}{2}\right) & x \in\left(-\frac{h}{2}, \frac{h}{2}\right) \\ u_{r} & x>\frac{h}{2},\end{cases}
$$

where $u_{l}, u_{r}, h>0$ are constants, and $a=\frac{1}{2}\left(u_{l}-u_{T}\right)>0$, and express $E$ in phaseamplitude form as $E=R \exp (i \theta)$. Let $\xi=x-c t$, where $c=\frac{1}{2}\left(u_{l}+u_{r}\right)$. The interval $(-h / 2+a t, h / 2-a t)$ at time $t=0$ is just $(-h / 2, h / 2)$; at time $t^{*}=h / 2 a$, it shrinks down to the point 0 . We call this interval the gap region, or just, 'gap'. With this notation, a solution of (4.0) can be given by

$$
\begin{align*}
& R(\xi, t)=\text { constant },  \tag{4.7}\\
& \theta(\xi, t)= \begin{cases}\theta_{\text {before }} & t<t^{*} \\
\theta_{\text {atter }} & t \geq t^{*},\end{cases}  \tag{4.8}\\
& u(\xi, t)= \begin{cases}u_{\text {before }} & t<t^{*} \\
u_{\text {after }} & t \geq t^{*},\end{cases} \tag{4.9}
\end{align*}
$$

where

$$
\begin{align*}
& \theta_{\text {before }}= \begin{cases}-u_{l} t & \xi<\text { gap } \\
\omega(t) \xi+\chi(t) & \xi \in \text { gap } \\
-u_{r} t & \xi>\text { gap }\end{cases}  \tag{4.10}\\
& \theta_{\text {arter }}= \begin{cases}-u_{l} t & \xi<0 \\
-u_{r} t & \xi>0\end{cases} \tag{4.11}
\end{align*}
$$

and

$$
\begin{align*}
& u_{\text {before }}= \begin{cases}u_{l} & \xi<\text { gap } \\
c-\frac{\xi}{t^{*}-t} & \xi \in \text { gap } \\
u_{r} & \xi>\mathrm{gap},\end{cases}  \tag{4.12}\\
& u_{\text {after }}= \begin{cases}u_{l} & \xi<0 \\
u_{r} & \xi>0 .\end{cases} \tag{4.13}
\end{align*}
$$

The coefficients $\omega(t)$ and $\chi(t)$ are given by

$$
\begin{align*}
\omega(t) & =-\log \left(t^{*}-t\right)  \tag{4.14}\\
\chi(t) & =\left(t^{*}-t\right)\left\{\log \left(t^{*}-t\right)\left[1+t^{*}-\frac{1}{2}\left(t^{*}-t\right)\right]-\frac{1}{4}\left(t^{*}-t\right)+t^{*}-1\right\}  \tag{4.15}\\
& -\left(\omega_{2}-c\right) t
\end{align*}
$$

(4.12) and (4.13) show that $u$ develops a shock at time $t=t^{*}$ and that this shock remains steady in the $\xi$ variable for $t>t^{*}$. Equation (4.14) implies that $|\omega(t)| \rightarrow \infty$ as $t \rightarrow t^{*}$. Thus system (4.0) has solutions in which shock formation in $u$ drives a corresponding blow-up in the traveling phase gradient, $\theta_{\xi}$, of the dispersive wave envelope. Furthermore, for $t>t^{*}, u(\xi, t)$ and $\theta(\xi, t)$ satisfy the equation

$$
\begin{equation*}
-\theta_{t}=u \tag{4.16}
\end{equation*}
$$

This equation will appear again in the small $\epsilon$ multi-scale expansion of (4.0). In the context of dispersive waves, the quantity $-\theta_{t}$ is interpreted as a generalized frequency
[2]. This is based on the simple observation that for a single linear mode, $\exp (i(k x-$ $\omega t)$ ), $-\theta_{t}$ is the usual frequency $\omega$. Equation (4.16) shows that, for the special solution constructed here, the frequency of the dispersive wave beyond break time has a moving discontinuity: to the left of $\xi=0$ it is $u_{l}$, and to the right of $\xi=0$ it is $u_{r}$.

## 5 The $\epsilon=0$ problem

When $\epsilon=0$, (4.0) reduces to

$$
\begin{align*}
& u\left(u+|E|^{2}\right)_{x}=0  \tag{5.1}\\
& i E_{t}+E_{x x}-u E=0 . \tag{5.2}
\end{align*}
$$

The two possible solutions of (5.1) are $u=0$ and $u=-|E|^{2}+c$, where $c$ may depend on $t$. We will denote either one of these solutions by $U_{0}$ and treat each $\epsilon=0$ problem separately. If $U_{0}=0$, equation (5.2) reduces to (4.1), the linear Schrödinger equation (LS) for $E$. When $U_{0}=-|E|^{2}+c$, equation (5.2) becomes

$$
\begin{equation*}
i E_{t}+E_{x x}+\left(|E|^{2}-c\right) E=0 \tag{5.3}
\end{equation*}
$$

which, for $c=0$, is the nonlinear Schrödinger equation (NLS). We will only be interested in the $c=0$ case. The perturbations of $u$ shall be of the form

$$
\begin{equation*}
u=u_{0}+\epsilon u_{1}+O\left(\epsilon^{2}\right), \tag{5.4}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{0}=U_{0}+v(x, t / \epsilon) . \tag{5.5}
\end{equation*}
$$

Note that the perturbation term $v(x, t / \epsilon)$ depends only on the fast time $t / \epsilon$. It is not assumed to be small in magnitude. The $\epsilon \rightarrow 0$ limit is singular in the sense that (i) $v(x, t / \epsilon)$ is not analytic in a neighborhood of $\epsilon=0$, and (ii) the magnitude of $v(x, t / \epsilon)$ need not go to zero with $\epsilon$, rather its wavespeed tends to infinity as $\epsilon$ goes to zero. If one thinks of $v(x, t / \epsilon)$ as representing an acoustic wave, the $\epsilon \rightarrow 0$ limit corresponds to the incompressible limit of the dispersive medium: as the compressibility of the medium
goes to zero, the sound wave's speed through that medium becomes infinite. The choice of initial data for $v$ depends on the phenomena we are interested in investigating. The Riemann data

$$
v(x, 0)=\left\{\begin{array}{ll}
\alpha & x \leq P  \tag{5.6}\\
0 & x>P
\end{array} \quad \alpha>0 \text { and } P \in \mathbf{R},\right.
$$

is the natural choice for the analysis of shock wave/dispersive wave interactions: it is the simplest data which leads to shock propagation. In the case $U_{0}=-|E|^{2}$, we shall focus on the interaction of a fast shock wave and a frozen soliton solution of NLS. When $U_{0}=0$, the soliton is replaced by a particular solution of LS which, in some ways, is a linear analogue of the soliton. Expressing $E$ in phase-amplitude form as $E=R \exp (i \theta)$, the two fundamental problems are stated as follows:

Problem 1. Find an asymptotic ( $\epsilon \rightarrow 0$ ) approximation to the solution of (4.0) such that

$$
\begin{align*}
& u_{0}=-R_{0}^{2}+v(x, t / \epsilon)  \tag{5.7}\\
& R \rightarrow 0 \text { as }|x| \rightarrow \infty \tag{5.8}
\end{align*}
$$

where $v(x, 0)$ is defined in (5.6), and $R_{0}$ is the leading term in the small $\epsilon$ expansion for the amplitude $R$.

Problem 2. Find an asymptotic ( $\epsilon \rightarrow 0$ ) approximation to the solution of (4.0) such that

$$
\begin{gather*}
u_{0}=0+v(x, t / \epsilon)  \tag{5.9}\\
R \rightarrow 0 \text { as }|x| \rightarrow \infty, \tag{5.10}
\end{gather*}
$$

where $v(x, 0)$ is defined in (5.6)

Note that for the singular Zakharov system [31, 32], the $\epsilon=0$ problem has the single solution $u=-|E|^{2}+c$ (modulo a linear function of $x$ ). As such, it is a perturbation only of NLS and not of LS.

## 6 The multi-scale expansions

To begin, rewrite system (4.0) in phase-amplitude form:

$$
\begin{align*}
& \epsilon u_{t}+u\left(u+R^{2}\right)_{x}=0  \tag{6.1}\\
& R_{t}+2 \theta_{x} R_{x}+\theta_{x x} R=0  \tag{6.2}\\
& R_{x x}-R\left(\theta_{t}+\left(\theta_{x}\right)^{2}+u\right)=0 . \tag{6.3}
\end{align*}
$$

The two time scales of the problem are given by

$$
\begin{array}{cl}
\tau_{1}=t / \epsilon, & \text { the fast time scale of the hyperbolic wave, } \\
\tau_{2}=t, & \text { the slow time scale of the dispersive wave. }
\end{array}
$$

For ease of notation, drop the subscripts and use $\tau$ and $t$ in place of $\tau_{1}$ and $\tau_{2}$, respectively. Rewriting (6.1)-(6.3) in the $x, t, \tau$ variables using the transformation

$$
\partial_{t} \rightarrow \partial_{t}+\epsilon^{-1} \partial_{\tau}
$$

gives the system

$$
\begin{align*}
& u_{T}+u\left(u+R^{2}\right)_{x}+\epsilon u_{t}=0  \tag{6.4}\\
& \epsilon^{-1} R_{\tau}+R_{t}+2 \theta_{x} R_{x}+\theta_{x x} R=0  \tag{6.5}\\
& \epsilon^{-1} \theta_{\tau}-R_{x x}+R\left(\theta_{t}+\left(\theta_{x}\right)^{2}+u\right)=0 . \tag{6.6}
\end{align*}
$$

Expand $u, R$, and $\theta$ in powers of $\epsilon$ :

$$
\begin{align*}
& u=u_{0}(x, t, \tau)+\epsilon u_{1}(x, t, \tau)+\cdots  \tag{6.7}\\
& R=R_{0}(x, t, \tau)+\epsilon R_{1}(x, t, \tau)+\cdots  \tag{6.8}\\
& \theta=\theta_{0}(x, t, \tau)+\epsilon \theta_{1}(x, t, \tau)+\cdots \tag{6.9}
\end{align*}
$$

substitute these expansions into equations (6.4)-(6.6), and collect like powers of $\epsilon$ through $O(\epsilon)$ to obtain the systems
$O\left(\frac{1}{6}\right):$

$$
\begin{align*}
& R_{0 \tau}=0  \tag{6.10}\\
& \theta_{0 \tau}=0 \tag{6.11}
\end{align*}
$$

$O(1):$

$$
\begin{align*}
& u_{0 \tau}+u_{0}\left(u_{0}+R_{0}^{2}\right)_{x}=0  \tag{6.12}\\
& R_{1 \tau}=-\left(R_{0 t}+2 R_{0 x} \theta_{0 x}+R_{0} \theta_{0 x x}\right)  \tag{6.13}\\
& \theta_{1 \tau}=\frac{1}{R_{0}} R_{0 x x}-\left(\theta_{0 t}+\left(\theta_{0 x}\right)^{2}+u_{0}\right) \tag{6.14}
\end{align*}
$$

$O(\epsilon):$

$$
\begin{align*}
& u_{1 \tau}+u_{0} u_{1 x}+\left(u_{0}+R_{0}^{2}\right)_{x} u_{1}=-2 u_{0}\left(R_{0} R_{1}\right)_{x}-u_{0 t}  \tag{6.15}\\
& R_{2 \tau}=-\left(R_{1 t}+2 R_{0 x} \theta_{1 x}+2 R_{1 x} \theta_{0 x}+R_{0} \theta_{1 x x}+R_{1} \theta_{0 x x}\right)  \tag{6.16}\\
& \theta_{2 \tau}=-\frac{R_{1}}{R_{0}} \theta_{1 \tau}+\frac{1}{R_{0}} R_{1 x x}-\left(\theta_{1 t}+2 \theta_{0 x} \theta_{1 x}+u_{1}\right)-\frac{R_{1}}{R_{0}}\left(\theta_{0 t}+\left(\theta_{0 x}\right)^{2}+u_{0}\right) . \tag{6.17}
\end{align*}
$$

### 6.1 Leading order shock equations

To derive the leading order shock equation for Problem 1, substitute (5.7) into (6.12). This gives

$$
\begin{align*}
& \frac{\partial v}{\partial \tau}+\left(v-R_{0}^{2}\right) \frac{\partial v}{\partial x}=0  \tag{6.18}\\
& v(x, 0)=f(x) \tag{6.19}
\end{align*}
$$

where $f(x)$ is the Riemann data (5.6). For Problem 2, substitute (5.9) into (6.12) to find

$$
\begin{align*}
& \frac{\partial v}{\partial \tau}+v \frac{\partial\left(v+R_{0}^{2}\right)}{\partial x}=0  \tag{6.20}\\
& v(x, 0)=f(x) \tag{6.21}
\end{align*}
$$

where $f(x)$ is the Riemann data (5.6).

Since $v$ must be independent of the slow time $t$ in equations (6.18) and (6.20), it is necessary to enforce the condition

$$
\begin{equation*}
\frac{\partial R_{0}}{\partial t}=0 . \tag{6.22}
\end{equation*}
$$

Since equations (6.10) and (6.11) state that $R_{0}$ and $\theta_{0}$ are independent of the fast time $\tau$, (6.22) implies that $R_{0}$ has only a spatial structure. Thus in the small $\epsilon$ limit, $R_{0}$ is frozen in time, which is reasonable in our approximation because the shock is moving so rapidly.

## 7 Modulation equations for the dispersive wave

For a description of the method of multiple time scales and many interesting applications of it, see [51]. The method is carried out by eliminating the $\tau$-independent, or secular, terms on the right-hand sides of equations (6.13), (6.14) and (6.16), (6.17). Setting the secular terms to zero gives the so-called modulation equations, or solvability conditions, for the amplitude and phase at orders $O(1)$ and $O(\epsilon)$. We derive the modulation equations for Problems 1 and 2 separately.

## 7.1 $O$ (1) modulation equations for Problem 1

Setting secular terms on the right-hand sides of equations (6.13) and (6.14) to zero and using equations (5.7), (6.22) gives the $O(1)$ modulation equations:

$$
\begin{align*}
& 2 \frac{d R_{0}}{d x} \frac{\partial \theta_{0}}{\partial x}+R_{0} \frac{\partial^{2} \theta_{0}}{\partial x^{2}}=0  \tag{7.1}\\
& \frac{1}{R_{0}} \frac{d^{2} R_{0}}{d x^{2}}-\frac{\partial \theta_{0}}{\partial t}-\left(\frac{\partial \theta_{0}}{\partial x}\right)^{2}+R_{0}^{2}=0 . \tag{7.2}
\end{align*}
$$

Equation (7.1) can be rewritten as

$$
\begin{equation*}
\frac{1}{R_{0}}\left(R_{0}^{2} \frac{\partial \theta_{0}}{\partial x}\right)_{x}=0 \tag{7.3}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\theta_{0}(x, t)=\alpha \int^{x} R_{0}^{-2}+\psi \tag{7.4}
\end{equation*}
$$

where $\alpha$ and $\psi$ may depend on $t$. Substituting (7.4) into (7.2) gives the leading order amplitude equation:

$$
\begin{equation*}
\frac{d^{2} R_{0}}{d x^{2}}-R_{0}\left(\dot{\alpha} \int^{x} R_{0}^{-2}+\dot{\psi}+\alpha^{2} R_{0}^{-4}-R_{0}^{2}\right)=0 \tag{7.5}
\end{equation*}
$$

Since $R_{0}$ is independent of $t$, we set $\dot{\alpha}=0$ and $\dot{\psi}=\lambda=$ constant in (7.5). Equations (7.4) and (7.5) then become

$$
\begin{align*}
& \theta_{0}(x, t)=\alpha \int_{0}^{x} R_{0}^{-2}+\lambda t+\text { constant }  \tag{7.6}\\
& \frac{d^{2} R_{0}}{d x^{2}}-R_{0}\left(\lambda+\alpha^{2} R_{0}^{-4}-R_{0}^{2}\right)=0 . \tag{7.7}
\end{align*}
$$

To focus on the interaction of a shock wave and a steady soliton, solve equations (7.6) and (7.7) by setting $\alpha=0$ and

$$
\begin{align*}
R_{0}^{2}(x) & =2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} x)  \tag{7.8}\\
\theta_{0}(t) & =\lambda t . \tag{7.9}
\end{align*}
$$

Equations (7.8), (7.9) represent the well-known single hump soliton solution of NLS [29].

## 7.2 $O$ (1) modulation equations for Problem 2

The only difference here is that $u_{0}$ satisfies equation (5.9) instead of (5.7). The resulting equations for $R_{0}$ and $\theta_{0}$ are

$$
\begin{align*}
& \theta_{0}(x, t)=\alpha \int_{0}^{x} R_{0}^{-2}+\lambda t+\text { constant }  \tag{7.10}\\
& \frac{d^{2} R_{0}}{d x^{2}}-R_{0}\left(\lambda+\alpha^{2} R_{0}^{-4}\right)=0 \tag{7.11}
\end{align*}
$$

The linear analogue of the soliton solution (7.8), (7.9) (again setting $\alpha=0$ ) is

$$
\begin{align*}
R_{0}^{2}(x) & =2 \lambda \exp (-2 \sqrt{\lambda}|x|)  \tag{7.12}\\
\theta_{0}(t) & =\lambda t . \tag{7.13}
\end{align*}
$$

The amplitude given by (7.12) is the Green's function solution of (7.11) with $\alpha=0$ [52]. It is continuous but has a jump discontinuity in its derivative at $x=0$. We shall


Figure 10: Amplitudes (7.8) and (7.12).
refer to (7.12), (7.13) as a 'linear soliton' since the shape of its amplitude is similar to (7.8) (Figure 10) and it decays to 0 as $|x| \rightarrow \infty$. This term is introduced only for convenience and is not meant to suggest any connections to the theory of integrable systems.

### 7.3 Generalized frequency of the dispersive wave

Once the $O$ (1) modulation equations are satisfied, equations (6.13) and (6.14) become

$$
\begin{align*}
\frac{\partial R_{1}(x, t, \tau)}{\partial \tau} & =0  \tag{7.14}\\
-\frac{\partial \theta_{1}(x, t, \tau)}{\partial \tau} & =v(x, \tau) \tag{7.15}
\end{align*}
$$

This holds for both Problems 1 and 2. Equation (7.14) states that $R_{1}$ is independent of the fast time $\tau$. Integrating (7.15) gives

$$
\begin{equation*}
\theta_{1}(x, t, \tau)=\int^{\tau} v+\bar{\theta}_{1}(x, t) \tag{7.16}
\end{equation*}
$$

The terms $\bar{\theta}_{1}$ and $R_{1}$ will be determined by the $O(\epsilon)$ modulation equations.
Compare equation (7.15) to equation (4.16) on page 46. In analogy with example 3 in section $4,-\theta_{1 r}$ is interpreted formally as the first correction term to the generalized frequency of the dispersive wave. In example 3, the steadily traveling shock wave consisted of the two constant states $u_{1}$ for $\xi<0, u_{r}$ for $\xi>0$, and the frequency $-\theta_{t}$
jumped from one to the other across $\xi=0$. Here, if the leading order term $v$ has a propagating shock separating two states, say $v_{l}(\tau)$ on the left and $v_{r}(\tau)$ on the right (which, in sections 8 and 9 , is shown to be the case), then (7.15) suggests that the first correction to the generalized frequency undergoes a sudden shift from $v_{l}$ to $v_{r}$ across the shock layer.

## 7.4 $O(\epsilon)$ modulation equations for Problem 1

With $u_{0}=-R_{0}^{2}+v$, equation (6.15) becomes

$$
\begin{equation*}
\frac{\partial u_{1}}{\partial \tau}+\left(v-R_{0}^{2}\right) \frac{\partial\left(u_{1}+2 R_{0} R_{1}\right)}{\partial x}+\frac{\partial v}{\partial x} u_{1}=0 . \tag{7.17}
\end{equation*}
$$

The change of variables

$$
\begin{equation*}
u_{1}=\bar{u}_{1}-2 R_{0} R_{1} \tag{7.18}
\end{equation*}
$$

transforms (7.17) to

$$
\begin{equation*}
\frac{\partial \hat{u}_{1}}{\partial \tau}+\left(v-R_{0}^{2}\right) \frac{\partial \hat{u}_{1}}{\partial x}+\frac{\partial v}{\partial x}\left(\hat{u}_{1}-2 R_{0} R_{1}\right)=0 . \tag{7.19}
\end{equation*}
$$

Substituting (7.6), (7.7), (7.16) and (7.18) into (6.16), (6.17) and setting $\tau$-independent terms on the right-hand sides to zero gives the general $O(\epsilon)$ modulation equations. They are

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(R_{0} R_{1}\right)+\frac{\partial}{\partial x}\left(R_{0}^{2} \frac{\partial \tilde{\theta}_{1}}{\partial x}+2 \alpha \frac{R_{1}}{R_{0}}\right)=0  \tag{7.20}\\
& \frac{\partial^{2} R_{1}}{\partial x^{2}}-\left(\lambda+\alpha^{2} R_{0}^{-4}-3 R_{0}^{2}\right) R_{1}=R_{0}\left(\frac{\partial \bar{\theta}_{1}}{\partial t}+2 \alpha R_{0}^{-2} \frac{\partial \tilde{\theta}_{1}}{\partial x}\right) \tag{7.21}
\end{align*}
$$

This is a linear system for $R_{1}(x, t)$ and $\bar{\theta}_{1}(x, t)$ with coefficients depending on $x$. Substituting the $O(1)$ soliton solution (7.8), (7.9) into (7.20), (7.21) gives

$$
\begin{align*}
& \frac{\partial R_{1}}{\partial t}+2 \frac{d R_{0}}{d x} \frac{\partial \bar{\theta}_{1}}{\partial x}+R_{0} \frac{\partial^{2} \tilde{\theta}_{1}}{\partial x^{2}}=0  \tag{7.22}\\
& \frac{\partial^{2} R_{1}}{\partial x^{2}}-\left(\lambda-3 R_{0}^{2}\right) R_{1}=R_{0} \frac{\partial \bar{\theta}_{1}}{\partial t} \tag{7.23}
\end{align*}
$$

To solve this system, we make the additional ansatz

$$
\begin{equation*}
\frac{\partial \bar{\theta}_{1}}{\partial x}=0 . \tag{7.24}
\end{equation*}
$$

Equation (7.22) then reduces to $R_{1 t}=0$, and (7.23) separates as

$$
\begin{align*}
& \frac{d^{2} R_{1}}{d x^{2}}-\left(\lambda-3 R_{0}^{2}\right) R_{1}=\beta R_{0}  \tag{7.25}\\
& \tilde{\theta}_{1}(t)=\beta t+\text { constant } \tag{7.26}
\end{align*}
$$

where $\beta$ is the constant of separation which we set to zero. We solve (7.25) by setting

$$
\begin{equation*}
R_{1}=\frac{d R_{0}}{d x} . \tag{7.27}
\end{equation*}
$$

Then (7.19) can be solved by setting

$$
\begin{equation*}
\hat{u}_{1}=\frac{\partial v}{\partial x}, \tag{7.28}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
u_{1}=\frac{\partial\left(-R_{0}^{2}+v\right)}{\partial x} \tag{7.29}
\end{equation*}
$$

When $v$ has shocks, (7.29) is interpreted in the small diffusion limit. It is well-known (see, for example, [46]) that by adding a diffusive term of the form $\eta v_{x x}$ to either of the leading order shock equations (6.18), (6.20), $v^{\eta}(x, \tau)$ is smoothed out. Replacing $v$ by $v^{\eta}$ when convenient still allows the qualitative aspects of the shock wave/dispersive wave interactions to come through. With this in mind, equation (7.29) is taken to mean

$$
\begin{equation*}
u_{1}=\frac{\partial\left(-R_{0}^{2}+v^{\eta}\right)}{\partial x} \tag{7.30}
\end{equation*}
$$

for small $\eta$.

## 7.5 $O(\epsilon)$ modulation equations for Problem 2

The equations for Problem 2 are similar to those of Problem 1. In this case, we do not make the change of variables (7.18). The general $O(\epsilon)$ modulation equations obtained
by substituting (7.10), (7.11) into (6.16), (6.17) are

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(R_{0} R_{1}\right)+\frac{\partial}{\partial x}\left(R_{0}^{2} \frac{\partial \tilde{\theta}_{1}}{\partial x}+2 \alpha \frac{R_{1}}{R_{0}}\right)=0  \tag{7.31}\\
& \frac{\partial^{2} R_{1}}{\partial x^{2}}-\left(\lambda+\alpha^{2} R_{0}^{-4}\right) R_{1}=R_{0}\left(\frac{\partial \tilde{\theta}_{1}}{\partial t}+2 \alpha R_{0}^{-2} \frac{\partial \tilde{\theta}_{1}}{\partial x}\right) \tag{7.32}
\end{align*}
$$

With the $O$ (1) solutions (7.12), (7.13), these reduce to

$$
\begin{align*}
& \frac{\partial R_{1}}{\partial t}+2 \frac{d R_{0}}{d x} \frac{\partial \bar{\theta}_{1}}{\partial x}+R_{0} \frac{\partial^{2} \bar{\theta}_{1}}{\partial x^{2}}=0  \tag{7.33}\\
& \frac{\partial^{2} R_{1}}{\partial x^{2}}-\lambda R_{1}=R_{0} \frac{\partial \bar{\theta}_{1}}{\partial t} \tag{7.34}
\end{align*}
$$

We again make the ansatz $\tilde{\theta}_{1 x}=0$. Equations (7.33), (7.34) are then separable, and can be solved as in Problem 1 by setting

$$
\begin{align*}
& R_{1}=\frac{d R_{0}}{d x}  \tag{7.35}\\
& \bar{\theta}_{1}=0 . \tag{7.36}
\end{align*}
$$

Equation (6.15) is then solved by setting

$$
\begin{equation*}
u_{1}=\frac{\partial v}{\partial x}, \tag{7.37}
\end{equation*}
$$

which is again interpreted as $u_{1}=\partial v^{\eta} / \partial x$ for a small diffusion coefficient $\eta$.

### 7.6 Expansion summary

For Problem 1 (page 48), we have

$$
\begin{align*}
& u \sim\left(-R_{0}^{2}(x)+v(x, \tau)\right)+\epsilon\left(-R_{0}^{2}(x)+v(x, \tau)\right)_{x}+O\left(\epsilon^{2}\right)  \tag{7.38}\\
& R \sim R_{0}+\epsilon\left(\frac{d R_{0}}{d x}\right)+O\left(\epsilon^{2}\right)  \tag{7.39}\\
& \theta \sim \lambda t-\epsilon\left(\int^{\tau} v\right)+O\left(\epsilon^{2}\right) \tag{7.40}
\end{align*}
$$

where $R_{0}^{2}(x)=2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} x)$, and $v(x, \tau)$ satisfies

$$
\begin{aligned}
& \frac{\partial v}{\partial T}+\left(v-R_{0}^{2}\right) \frac{\partial v}{\partial x}=0 \\
& v(x, 0)=\text { Riemann data (5.6). }
\end{aligned}
$$

For Problem 2,

$$
\begin{align*}
& u \sim v(x, \tau)+\epsilon\left(v_{x}(x, \tau)\right)+O\left(\epsilon^{2}\right)  \tag{7.41}\\
& R \sim R_{0}+\epsilon\left(\frac{d R_{0}}{d x}\right)+O\left(\epsilon^{2}\right)  \tag{7.42}\\
& \theta \sim \lambda t-\epsilon\left(\int_{v}^{\tau}\right)+O\left(\epsilon^{2}\right), \tag{7.43}
\end{align*}
$$

where $R_{0}^{2}(x)=2 \lambda \exp (-2 \sqrt{\lambda}|x|)$, and $v(x, \tau)$ satisfies

$$
\begin{aligned}
& \frac{\partial v}{\partial \tau}+v \frac{\partial\left(v+R_{0}^{2}\right)}{\partial x}=0 \\
& v(x, 0)=\text { Riemann data (5.6). }
\end{aligned}
$$

To understand the leading order interactions, it is thus necessary to analyze $v$. We address this issue for Problems 1 and 2 in the next two sections.

## 8 Solution of the shock equation for Problem 1

In this section the exact solution of equations (6.18), (6.19) is presented. The solution includes a detailed description of the shock structure. We think of (6.18) as a conservation law with a spatially dependent wavespeed given by $v-R_{0}^{2}(x)$. The equation may be thought of as a conservation law governing a shock propagating through an inhomogeneous medium [16]. We shall show that the the shock speed is reduced as the shock approaches the hump of the soliton, and if the soliton is large enough, the speed may go to zero. Furthermore, if the shock starts in a particular interval centered about the peak of the soliton, its direction mat be reversed.

We first consider the characteristic equations derived by rewriting (6.18), (6.19) as a system of ode's for $v(x(s), \tau(s))$. The system is

$$
\frac{d v}{d s}=0, \frac{d x}{d s}=v-R_{0}^{2}(x(s)), \frac{d \tau}{d s}=1,
$$

with initial conditions

$$
\tau(0)=0, x(0)=\xi, v(0)=f(\xi)
$$

where $f(x)$ is the Riemann data (6.19), and $R_{0}^{2}$ is given by (7.8). These equations are equivalent to the system

$$
\begin{align*}
v & =f(\xi)  \tag{8.1}\\
\frac{d x}{d \tau} & =f(\xi)-R_{0}^{2}(x(\tau))  \tag{8.2}\\
x(0) & =\xi . \tag{8.3}
\end{align*}
$$

The characteristic curve emanating from the point $(\xi, 0)$ will be denoted by $x_{\xi}(\tau)$. Equation (8.2) can be written as

$$
\begin{equation*}
\frac{d \tau}{d x}=\frac{1}{f(\xi)-R_{0}^{2}(x)}, \tag{8.4}
\end{equation*}
$$

and in this form, the exact solution $\tau(x ; \xi)$ can be expressed in closed form. Since the exact formulas are not needed, we do not give them here. The shock structure for equation (6.18) with data (5.6) is well known in the case $\lambda=0$, i.e., in the absence of a dispersive wave. The curves $x_{\xi}(\tau)$ are straight lines of slope $\alpha$ for $\xi \leq P$, vertical lines for $\xi>P$, and the shock curve $s(\tau)$ is a straight line of slope $\alpha / 2$ (Figure 11). This means that the discontinuity simply travels to the right at the constant speed $\alpha / 2$. Figure 13 on page 65 graphs the characteristic curves for various values of $\lambda$. The curves in the figure are obtained by exact integration of equations (8.1)-(8.3). The following lemma shows that for the Riemann data (6.19), no new shocks develop beyond $\tau=0$.

Lemma 3. There exists a unique shock curve $s(\tau)$ in the characteristic plane. It starts at the point $(P, 0)$.


Figure 11: Straight characteristics impinging upon the shock curve $s(\tau)$ in the absence of a dispersive wave.

Proof. It suffices to show that characteristic curves emanating from either side of $P$ do not cross. If $\xi_{1}<\xi_{2} \leq P$, then equations (8.1)-(8.3) for the characteristic curves $x_{\xi_{1}}(\tau)$ and $x_{\xi_{2}}(\tau)$ are

$$
\begin{align*}
\frac{d x_{\xi_{1}}}{d \tau} & =\alpha-R_{0}^{2}\left(x_{\xi_{1}}\right)  \tag{8.5}\\
x_{\xi_{1}}(0) & =\xi_{1} \tag{8.6}
\end{align*}
$$

and

$$
\begin{align*}
\frac{d x_{\xi_{2}}}{d \tau} & =\alpha-R_{0}^{2}\left(x_{\xi_{2}}\right)  \tag{8.7}\\
x_{\xi_{2}}(0) & =\xi_{2} \tag{8.8}
\end{align*}
$$

In the $(\tau, x)$ plane, the curves, when considered as functions $\tau(x)$, are vertical translates of each other and thus can not intersect. That is, each curve is of the form $\tau(x)-\tau(\xi)$, where $\tau(x)$ solves (8.4). The same holds for curves emanating from the right of $P$.

### 8.1 Solution of the Riemann problem

Theorem 2. The solution of equation (6.18) with data (6.19) is

$$
v(x, \tau)= \begin{cases}\alpha & x \leq s(\tau)  \tag{8.9}\\ 0 & x>s(\tau)\end{cases}
$$

where $s(\tau)$ is described in Theorem 3.

Proof. Equation (8.1) states that $v$ is constant on characteristic curves. At any point $(x, \tau), v$ is thus determined by tracing the characteristic curve through $(x, \tau)$ backwards in time to the $\tau=0$ axis. Points to the left of the shock trace back to the value $\alpha$, and points to the right of the shock trace back to the value 0 .

### 8.2 Shock structure for Problem 1

The shock motion in the presence of the soliton depends on where the shock starts, given by $P$, and the ratio of the unperturbed shock speed $\alpha / 2$ to the soliton amplitude $2 \lambda$. To summarize,

- The shock either travels right for all time or left for all time, depending on where it starts in relation to the center of the soliton.
- Depending on the size of $\alpha / 2$ compared to $2 \lambda$, the shock may be trapped in either direction, or it may escape to infinity in the forward direction.
- If the shock escapes, its speed converges to $\alpha / 2$.

We thus see that the shock front may be completely blocked by the soliton if the soliton is large enough (or, equivalently, if the shock strength is weak enough). Furthermore, the influence of the soliton is localized in the sense that if the shock passes through the soliton, it recovers its unperturbed speed of $\alpha / 2$, the propagation speed in the absence of a dispersive wave. We will see that this is not the case for the leading order shock equation of Problem 2. There, the effect of the soliton on the long term shock speed can, in some cases, be permanent. Figures 18,19 , and 20 show numerical solutions of equation (6.18) which demonstrate the phenomena described in the following theorem.

Theorem 3 (Shock dynamics).
Case $\alpha<4 \lambda$. Let $N_{1}<0<N_{2}$ be the zeros of $\alpha / 2-2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} x)$.
Forwards capture: If $P<N_{\mathrm{l}}$, the shock location increases to $N_{\mathrm{l}}$ as $\tau \rightarrow \infty$ (Figure 19).

Backwards capture: If $N_{1}<P<N_{2}$, the shock location decreases to $N_{1}$ as $\tau \rightarrow \infty$ (Figure 18).

Escape: If $P>N_{2}$, the shock location increases to infinity as $\tau \rightarrow \infty$, and its speed converges to $\alpha / 2$ (Figure 20).
Steady state: If $P=N_{1}$ or if $P=N_{2}$, the shock location does not move.
Case $\alpha=4 \lambda$.
Forwards capture: If $P<0$, the shock location increases to 0 as $\tau \rightarrow \infty$.
Escape: If $P>0$, the shock location increases to infinity as $\tau \rightarrow \infty$, and its
speed converges to $\alpha / 2$.
Steady state: If $P=0$, the shock location does not move.
Case $\alpha>4 \lambda$.
Escape: For any $P$, the shock location increases to infinity as $\tau \rightarrow \infty$, and its speed converges to $\alpha / 2$.

Proof. We first derive the Rankine-Hugoniot condition which governs the motion of the shock. In [53], it is shown that for an equation of the form

$$
\begin{equation*}
U_{\tau}+U U_{x}=a(x) U, \tag{8.10}
\end{equation*}
$$

the shock curve $S(\tau)$ satisfies

$$
\begin{equation*}
\frac{d S}{d \tau}=\frac{1}{2}\left(U^{-}+U^{+}\right) \tag{8.11}
\end{equation*}
$$

where

$$
\begin{equation*}
U^{ \pm}(\tau)=\lim _{x \rightarrow S(\tau)^{ \pm}} U(x, \tau) . \tag{8.12}
\end{equation*}
$$

In equation (6.18), if we set $U=v-R_{0}^{2}$ then the shock location, $S(\tau)$, for $U$ is the same as the shock location, $s(\tau)$, for $v$. Equation (6.18) in the $U$ variable is (8.10) with
$a(x)=-\frac{d}{d x} R_{0}^{2}$. Since $v^{-}=\alpha$ and $v^{+}=0$, then $U^{-}=\alpha-R_{0}^{2}(s)$ and $U^{+}=0-R_{0}^{2}(s)$.
Therefore

$$
\frac{d s}{d \tau}=\frac{1}{2}\left(U^{-}+U^{+}\right)=\frac{1}{2}\left(\alpha-2 R_{0}^{2}(s)\right)=\frac{\alpha}{2}-R_{0}^{2}(s)
$$

Since $s$ starts at $P$, the equations for $s(\tau)$ are simply

$$
\begin{align*}
\frac{d s}{d \tau} & =\frac{\alpha}{2}-R_{0}^{2}(s)  \tag{8.13}\\
s(0) & =P . \tag{8.14}
\end{align*}
$$

The dynamics of $s(\tau)$ as described in Theorem 3 follow from the $(s, s)$ phase-plane diagrams below (Figure 12). In each diagram, the arrows indicate the direction the shock moves. Figure 12a shows the case $\alpha<4 \lambda$. There are fixed points at $N_{1}$ and $N_{2}$. If $s(0) \in\left(N_{1}, N_{2}\right)$, then $d s / d \tau<0$ and $s \rightarrow N_{1}$. If $P \notin\left(N_{1}, N_{2}\right), d s / d \tau>0$, so $s \rightarrow N_{1}$ if $s(0)<N_{1}$, and $s \rightarrow \infty$ if $s(0)>N_{2}$. The $\alpha=4 \lambda$ case is shown in Figure 12b. Here the interval ( $N_{1}, N_{2}$ ) collapses to the point at the origin, so if $s(0)<0, s$ increases to 0 , if $s(0)=0$, then $s$ stays at 0 , and if $s(0)>0$, then $s \rightarrow \infty$. Figure 12c shows the $\alpha>4 \lambda$ case. In this case, $d s / d \tau$ is always positive, and no equilibrium points exist, hence $s \rightarrow \infty$. Each phase portrait is essentially the graph of $\alpha / 2-2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} s)$, which approaches $\alpha / 2$ as $|s| \rightarrow \infty$. Thus the shock speed approaches $\alpha / 2$ if the shock escapes.

## 9 Solution of the shock equation for Problem 2

We now consider equations (6.20), (6.21). Although it is not possible to write down an exact formula for the solution as we did for Problem 1, there are exact formulas for the long-time solution profile which make the effect of the dispersive wave on the shock wave apparent. We think of equation (6.20) as a conservation law with a spatially varying friction term $-v\left(d R_{0}^{2} / d x\right)$ due to the dispersive wave amplitude [7], which in this case is the linear analogue of the soliton. For Problem 1, we saw that $v$ retained the shape of the initial data as the discontinuity propagated in fast time. For Problem


Figure 12: ( $s, \dot{s}$ ) phase plane. (a) $\alpha<4 \lambda$. (b) $\alpha=4 \lambda$. (c) $\alpha>4 \lambda . N_{1}$ and $N_{2}$ are the zeros of $\alpha / 2-2 \lambda \operatorname{sech}^{2}\left(\lambda^{1 / 2} s\right)$.

2, though $v$ remains positive, it may increase or decrease depending on the sign of the friction term. If the linear soliton is large enough, it may push $v$ down to zero thereby reducing the shock speed to zero. In section 10 we show some numerical solutions of (6.20) (6.21) (with an added small diffusive term) that give a clear picture of how $v$ evolves and how the long-time states described in this section are obtained. The analysis is again based on the characteristic equations.

### 9.1 Characteristic equations

The characteristic equations for (6.20), (6.21) are

$$
\frac{d v}{d s}=-v \frac{d}{d x}\left(R_{0}^{2}\right), \frac{d x}{d s}=v, \frac{d \tau}{d s}=1
$$

with initial conditions

$$
\tau(0)=0, x(0)=\xi, \quad v(0)=f(\xi)
$$

where $f(x)$ is the Riemann data (5.6) and $R_{0}^{2}(x)$ is given by (7.12). These equations are equivalent to the system

$$
\begin{align*}
& v=f(\xi)+R_{0}^{2}(\xi)-R_{0}^{2}(x(\tau))  \tag{9.1}\\
& \frac{d x}{d \tau}=f(\xi)+R_{0}^{2}(\xi)-R_{0}^{2}(x(\tau))  \tag{9.2}\\
& x(0)=\xi \tag{9.3}
\end{align*}
$$

The phase plane ( $x_{\xi}, d x_{\xi} / d \tau$ ) for each characteristic curve now depends on the parameter $\xi$, in contrast to the situation in Problem 1. In this sense system (9.1)-(9.3) is nonautonomous. If (9.2) is written as

$$
\begin{equation*}
\frac{d \tau}{d x}=\frac{1}{f(\xi)+R_{0}^{2}(\xi)-R_{0}^{2}(x)} \tag{9.4}
\end{equation*}
$$

the exact solution $\tau(x ; \xi)$ is expressible in closed form. Since the closed form solution is not needed, we do not write it here. Figure 13c graphs the characteristic curves for various values of $\lambda$. The curves in the figure are obtained by exact integration of equations (9.1)-(9.3).


Figure 13: Comparison of characteristic curves. (a) Initial data (5.6) with $\alpha=1$. (b) Problem 1. Solid: $\lambda=0$, small dash: $\lambda=0.1$, large dash: $\lambda=0.3$. (c) Problem 2. Solid: $\lambda=0$, small dash: $\lambda=0.3$, large dash: $\lambda=0.7$.

With data (5.6), the characteristic equations become

$$
\begin{array}{rlr}
v & =\alpha+R_{0}^{2}(\xi)-R_{0}^{2}(x(\tau)) \\
\frac{d x}{d \tau} & =\alpha+R_{0}^{2}(\xi)-R_{0}^{2}(x(\tau)) & \\
x(0) & =\xi \\
& & \\
v & =R_{0}^{2}(\xi)-R_{0}^{2}(x(\tau)) \\
\frac{d x}{d \tau} & =R_{0}^{2}(\xi)-R_{0}^{2}(x(\tau))  \tag{9.10}\\
x(0) & =\xi & \\
\end{array}
$$

Denoting by $x_{\xi}(\tau)$ the characteristic curve from the point $(\xi, 0)$, the solution of the $\xi>P$ system is simply

$$
\begin{align*}
v & =0  \tag{9.11}\\
x_{\xi}(\tau) & =\xi, \tau \geq 0 . \tag{9.12}
\end{align*}
$$

Thus for $\xi>P$, characteristic curves are vertical lines along which $v=0$. For $\xi \leq P$, the curves $x_{\xi}(\tau)$ are not straight, but have initial slope $\alpha$ since

$$
\begin{align*}
\left.\left(\frac{d x_{\xi}}{d \tau}\right)\right|_{\tau=0} & =\left.\left(\alpha+R_{0}^{2}(\xi)-R_{0}^{2}\left(x_{\xi}(\tau)\right)\right)\right|_{\tau=0}  \tag{9.13}\\
& =\alpha+R_{0}^{2}(\xi)-R_{0}^{2}(\xi) \\
& =\alpha
\end{align*}
$$

The next lemma describes the long-time behavior of the characteristic curves originating to the left of $P$.

Lemma 4. Suppose $\xi \leq P$. If $\alpha \geq 2 \lambda$, the curves $x_{\xi}(\tau)$ increase (travel right) to infinity as $\tau \rightarrow \infty$. If $\alpha<2 \lambda$, define $\Omega$ by

$$
\begin{equation*}
\Omega<0, \quad \alpha+R_{0}^{2}(\Omega)=2 \lambda, \tag{9.14}
\end{equation*}
$$

and for $\xi \leq \Omega$, define

$$
\begin{equation*}
A_{\xi}=\frac{1}{2 \sqrt{\lambda}} \log \left(\frac{\alpha+R_{0}^{2}(\xi)}{2 \lambda}\right)<0 \tag{9.15}
\end{equation*}
$$

Then for $\xi \leq \Omega, x_{\xi}(\tau)$ increases to a vertical asymptote at $A_{\xi}$, while for $\xi>\Omega, x_{\xi}(\tau)$ increases to infinity as $\tau \rightarrow \infty$.

Proof. If $\alpha \geq 2 \lambda$, then for a fixed point $\xi, \alpha+R_{0}^{2}(\xi)>2 \lambda+\epsilon$ where $\epsilon>0$ depends on the point $\xi$. Then

$$
\frac{d x_{\xi}}{d \tau}=\alpha+R_{0}^{2}(\xi)-R_{0}^{2}(x)>\epsilon
$$

and so $x_{\xi}(\tau)$ increases to infinity as $\tau \rightarrow \infty$. This is true for each $\xi$.
For $\alpha<2 \lambda$, refer to the $\left(x_{\xi}, \dot{x}_{\xi}\right)$ phase planes shown in Figure 14. If $\xi<\Omega$ (Figure 14a), there are equilibrium points at $A_{\xi}$ and $-A_{\xi}$. Since $\Omega<A_{\xi}$, then $x_{\xi}(\tau)$ increases to $A_{\xi}$ as $\tau \rightarrow \infty$. If $\xi=\Omega$ (Figure 14 b ), the equilibria coalesce at the origin: $\pm A_{\xi}=A_{\Omega}=0$. Since $\Omega<0$, then $x_{\xi}(\tau)$ increases to 0 as $\tau \rightarrow \infty$. If $\Omega<\xi<-\Omega$ (Figure 14 c ), then $R_{0}^{2}(\xi)>2 \lambda-\alpha$, so there are no equilibrium points, and $x_{\xi}(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. If $\xi \geq-\Omega$ (Figure 14d), there are equilibria at $\pm A_{\xi}$, but since $x_{\xi}(\tau)$ starts to the right of these equilibria, $x_{\xi}(\tau)$ increases to infinity.

Note that the asymptote location $A_{\xi}$ increases with $\xi$ on $(-\infty, \Omega]$ and satisfies

$$
\begin{equation*}
\frac{1}{2 \sqrt{\lambda}} \log \left(\frac{\alpha}{2 \lambda}\right)<A_{\xi} \leq 0 \tag{9.16}
\end{equation*}
$$

This means that if $\varepsilon_{1}<\varepsilon_{0}$ then $x_{f}(\tau)$ asvmototes before $x_{\varepsilon_{n}}(\tau)$, and that all the asympanptote locations are within the interval $\left(\frac{1}{2 \sqrt{\lambda}} \log \left(\frac{\alpha}{2 \lambda}\right), 0\right]$. We shall see later that under certain conditions on $P$, this is the interval in which the shock can stop.

### 9.2 Single or multiple shocks

Regions where characteristic curves cross correspond to regions where $v$ would be multivalued. Shock curves prevent the crossing of characteristic curves [1, 2]. In this section we show that if $P \leq 0$, there is only one shock, and if $P>0$ is sufficiently large, a transient shock develops at some positive time. In section 10, numerical examples are presented which show the development of the transient shock when $P$ is large enough (Figure 21) and the suppression of the transient shock for $P$ near 0 (Figure 22).

## (a) <br> 

(b)

(c)

(d)


Figure 14: $\left(x_{\xi}, \dot{x}_{\xi}\right)$ phase planes for $\alpha<2 \lambda$.
(a) $\xi<\Omega$. (b) $\xi=\Omega$. (c) $\xi \in(\Omega,-\Omega)$. (d) $\xi \geq-\Omega$.


Figure 15: Problem 2 pullbacks. $\xi_{s}^{-}$is the left pullback; it decreases in time. The right pullback, $\xi_{s}^{+}$, increases in time.

Lemma 5. The Rankine-Hugoniot condition for a shock curve $s(\tau)$ is

$$
\begin{equation*}
\frac{d s}{d \tau}=\frac{1}{2}\left(v^{-}+v^{+}\right) \tag{9.17}
\end{equation*}
$$

where $v^{ \pm}$are defined in equation (8.12).

Proof. Since (6.20) has the same form as (8.10), then equations (8.11) and (8.12) hold with $U$ replaced by $v$.

By equation (9.1), the terms $v^{ \pm}$in (9.17) are

$$
\begin{align*}
& v^{-}=f\left(\xi_{s}^{-}\right)+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(s)  \tag{9.18}\\
& v^{+}=f\left(\xi_{s}^{+}\right)+R_{0}^{2}\left(\xi_{s}^{+}\right)-R_{0}^{2}(s), \tag{9.19}
\end{align*}
$$

where $\xi_{s}^{-}\left(\xi_{s}^{+}\right)$is the left (right) pullback from the shock curve to the $\tau=0$ axis. That is, to find $\xi_{z}^{-}\left(\xi_{s}^{+}\right)$, follow the left (right) impinging characteristic curve down to where it originates (Figure 15). Lemma 4 shows that characteristic curves always have positive slope. By equation (9.1), this implies that $v$ is always positive, hence $v^{ \pm}$are always positive. Therefore, shocks always travel to the right. Because of this, $\xi_{s}^{-}$must decrease as $\tau$ increases, and $\xi_{s}^{+}$must increase with $\tau$.


Figure 16: The characteristic curves $x_{\xi_{1}}(\tau)$ and $x_{\xi_{2}}(\tau)$. If $P \leq 0$, then $x_{\xi_{2}}(\tau)$ is always flatter and lower than $x_{\xi_{1}}(\tau)$.

Theorem 4. If $P \leq 0$, the shock curve $s(\tau)$ starting at $(P, 0)$ is the only shock that occurs.

Proof. Since characteristic curves to the right of $P$ are vertical lines, they clearly do not intersect each other. Let $\xi_{1}<\xi_{2} \leq P$ (Figure 16). We show that $x_{\xi_{1}}(\tau)$ can not intersect $x_{\xi_{2}}(\tau)$. This will prove that one shock curve suffices to prevent all crossing of characteristics. The curve $\tau_{\xi_{1}}(\tau)$ starts at the point $\left(\xi_{1}, 0\right)$ in the $(x, \tau)$ plane. If it has a vertical asymptote before $\xi_{2}$, i.e., if $A_{\xi_{1}} \leq \xi_{2}$, then it can not intersect $x_{\xi_{2}}(\tau)$. Suppose that either $x_{\xi_{1}}(\tau)$ has no vertical asymptote or that $A_{\xi_{1}}>\xi_{2}$. When $x_{\xi_{1}}(\tau)$ increases to $\xi_{2}$, it is above the point $\left(\xi_{2}, 0\right)$, the starting point for the curve $x_{\xi_{2}}(\tau)$. Comparing the slopes of the curves at any point $x \geq \xi_{2}$ gives

$$
\begin{equation*}
\left.\frac{d}{d \tau}\right|_{x} x_{\xi_{1}}(\tau)=\alpha+R_{0}^{2}\left(\xi_{1}\right)-R_{0}^{2}(x)<\alpha+R_{0}^{2}\left(\xi_{2}\right)-R_{0}^{2}(x)=\left.\frac{d}{d \tau}\right|_{x} x_{\xi_{2}}(\tau), \tag{9.20}
\end{equation*}
$$

where the inequality comes from the fact that $R_{0}^{2}\left(\xi_{1}\right)<R_{0}^{2}\left(\xi_{2}\right)$. Since $x_{\xi_{2}}(\tau)$ starts below $x_{\xi_{1}}(\tau)$ and is always flatter than $x_{\xi_{1}}(\tau)$, the curves can not intersect.

Theorem 5. Suppose $P>0$. If $P$ is sufficiently large, in addition to the shock $s(\tau)$ starting at $(P, 0)$, there is a second, transient shock $\bar{s}(\tau)$ which starts at some time
$\tau^{*}>0$ and merges with $s(\tau)$ in finite time.
Figure 21 on page 86 shows the development of the transient shock.
Proof. If $0 \leq \xi_{1}<\xi_{2} \leq P$, then since $R_{0}^{2}$ is decreasing on ( $0, \infty$ ) we have

$$
\begin{equation*}
\left.\frac{d}{d \tau}\right|_{x} x_{\xi_{1}}(\tau)=\alpha+R_{0}^{2}\left(\xi_{1}\right)-R_{0}^{2}(x)>\alpha+R_{0}^{2}\left(\xi_{2}\right)-R_{0}^{2}(x)=\left.\frac{d}{d \tau}\right|_{x} x_{\xi_{2}}(\tau) . \tag{9.21}
\end{equation*}
$$

Thus the curve $x_{\xi_{1}}(\tau)$ is always flatter than $x_{\xi_{2}}(\tau)$. Since $x_{\xi_{1}}(\tau)$ starts to the left of $x_{\xi_{2}}(\tau)$, the curves will eventually intersect. This holds for all characteristic curves starting between 0 and $P$. If $P$ is close to 0 , the shock $s(\tau)$ may prevent the crossing of these characteristic curves. But if $P$ is sufficiently large, a new shock $\tilde{s}$ must form to prevent crossing.

To prove that $\bar{s}$ merges with $s$, we consider the right pullbacks $\tilde{\xi}^{+}$of the transient shock, and the left pullbacks $\xi^{-}$of the permanent shock. The right pullbacks increase in time, while the left pullbacks decrease in time. We show that they converge to the same point. Since all characteristic curves emanating from the right of $P$ must hit the shock $s(\tau)$, we have that $\bar{\xi}^{+} \leq P$. Since $\bar{\xi}^{+}$is increasing and bounded above by $P$, then $\tilde{\xi}^{+} \rightarrow \bar{k}$ for some $\bar{k} \leq P$. Likewise, since $\xi^{-}$is decreasing and bounded below by $0, \xi^{-} \rightarrow k$ for some $k \geq 0$. We certainly have $\bar{k} \leq k$. We show that equality holds. If not, we could pick a point $b$ with $\vec{k}<b<k$ and consider the characteristic curve $x_{b}(\tau)$. It could not impinge on either shock curve. Its slope at a point $x$ is $\alpha+R_{0}^{2}(b)-R_{0}^{2}(x)$. But since $b<k<\xi^{-}$, then

$$
\begin{align*}
\alpha+R_{0}^{2}(b)-R_{0}^{2}(x) & >\alpha+R_{0}^{2}(k)-R_{0}^{2}(x)  \tag{9.22}\\
& >\alpha+R_{0}^{2}(\xi)-R_{0}^{2}(x)  \tag{9.23}\\
& >\frac{1}{2}\left(\alpha+R_{0}^{2}(\xi)-R_{0}^{2}(x)\right)  \tag{9.24}\\
& =\left.\frac{d s}{d \tau}\right|_{s=x} . \tag{9.25}
\end{align*}
$$

This implies that $x_{b}(\tau)$ would eventually intersect $s(\tau)$, which is a contradiction. Therefore, we must have $\bar{k}=k$. Since $x_{k}(\tau)$ impinges on both $s$ and $\bar{s}$, the shocks must intersect. After their intersection, the single shock $s(\tau)$ prevents any further crossing of characteristics.

### 9.3 Long-time dynamics

In the discussion following the proof of Lemma 4, we noted that $d s / d \tau$ is always positive. This implies that $s(\tau)$ either increases to infinity or to a finite supremum. We call the former case shock escape and the latter case shock capture. The analysis is divided into two parameter regimes: $\alpha \geq 2 \lambda$ and $\alpha<2 \lambda$. In each case, numerical experiments and analytical arguments are used to describe when shock capture or escape occurs. Exact formulas are given for the long-time structure of the solution for various ranges of the parameters $\alpha, \lambda$ and $P$. In short, there is no shock capture for $\alpha>2 \lambda$, while for $\alpha<2 \lambda$ there can be shock capture or escape depending on $P$.

## Case: $\alpha>2 \lambda$

Theorem 6 (Shock escape-no speed change). If $\alpha>2 \lambda$, the shock passes through the linear soliton and goes to infinity as $\tau \rightarrow \infty$. The shock speed converges to $\alpha / 2$, which is the shock speed for equation (6.20) in the absence of a dispersive wave.

A numerical example demonstrating Theorem 6 is shown in Figure 23 on page 90.

Proof. Let $\alpha-2 \lambda=\epsilon>0$. The slope of the shock curve satisfies

$$
\begin{equation*}
\frac{d s}{d \tau}=\frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(s)\right)>\frac{1}{2}\left(\alpha-R_{0}^{2}(s)\right)>\epsilon . \tag{9.26}
\end{equation*}
$$

For $s$ to converge to a finite point, its slope would have to converge to zero, which (9.26) shows is not possible.

To find the limiting slope of $s$, we consider the left pullbacks $\xi_{s}^{-}(\tau)$. The left pullbacks are decreasing in time, so go either to minus infinity or to a finite infimum. We claim that they go to minus infinity. For suppose $\xi_{s}^{-}$decreased to an infimum $k>-\infty$. By Lemma 4, we know that the characteristic curve $x_{k}(\tau)$ increases to infinity. By (9.2), its slope at a point $x$ is

$$
\begin{equation*}
\frac{d x_{k}}{d \tau}=\alpha+R_{0}^{2}(k)-R_{0}^{2}(x) \tag{9.27}
\end{equation*}
$$

which goes to $\alpha+R_{0}^{2}(k)$ as $x \rightarrow \infty$. The slope of the shock curve is

$$
\begin{equation*}
\frac{d s}{d \tau}=\frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(s)\right) \tag{9.28}
\end{equation*}
$$

which goes to $\frac{1}{2}\left(\alpha+R_{0}^{2}(k)\right)$ as $s \rightarrow \infty$. This implies that $x_{k}(\tau)$ eventually intersects $s(\tau)$. Any point on the shock beyond where $x_{k}(\tau)$ impinges upon it will have a left pullback less than $k$. This contradicts the fact that $k$ is the infimum over all left pullbacks. Thus $\xi_{s}^{-} \rightarrow-\infty$ as $\tau \rightarrow \infty$. Therefore,

$$
\frac{d s}{d \tau}=\frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(s)\right) \rightarrow \alpha / 2
$$

which completes the proof.
Theorem 7. The long-time structure of the solution of (6.20), (6.21) when $\alpha>2 \lambda$ is given by the formula

$$
v(x, \tau) \rightarrow \begin{cases}\alpha-R_{0}^{2}(x) & x \leq s(\tau)  \tag{9.29}\\ 0 & x>s(\tau)\end{cases}
$$

Figure 17 a shows the long-time profile (9.29).
Proof. By (9.11) and (9.12), $v$ is always zero to the right of the shock. Pick a point $X<s(\tau)$. The left pullbacks $\xi_{X}(\tau)$ of the vertical line $x=X$ decrease to minus infinity. If not, they would decrease to a finite infimum, $k$, and by looking at the characteristic curve $x_{k}(\tau)$, we could derive a contradiction. By (9.5), we have

$$
\lim _{\tau \rightarrow \infty} v(X, \tau)=\lim _{\tau \rightarrow \infty} \alpha+R_{0}^{2}\left(\xi_{\bar{X}}\right)-R_{0}^{2}(X)=\alpha-R_{0}^{2}(X)
$$

This proves the theorem.

The proofs of Theorems 6 and 7 go through when $\alpha=2 \lambda$ provided we restrict to the case $P \geq 0$. The only alteration required is in proving that the shock can not stop at a finite point $X$. This is done as follows. Since $s(0)=0$ and $\dot{s}(0)=\alpha / 2$, then beyond some time $\tau_{0}>0$, we know that $s \geq \delta$ for some $\delta>0$. Then since $R_{0}^{2}$ is decreasing on
$(0, \infty)$ and $R_{0}^{2}(0)=\alpha$, then $\alpha-R_{0}^{2}(s) \geq \epsilon>0$ for all $\tau>\tau 0$. As a result, the shock speed at any finite point $X$ satisfies

$$
\begin{equation*}
\frac{d s}{d \tau}=\frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(s)\right)>\frac{1}{2}\left(\alpha-R_{0}^{2}(s)\right) \geq \epsilon \tag{9.30}
\end{equation*}
$$

for $\tau>\tau_{0}$. This shows that the shock can not stop at any finite point $X$. Numerical evidence suggests that Theorems 6 and 7 extend to the case $\alpha=2 \lambda$ and $P<0$, though no proof of this is currently available.

Case: $\alpha<2 \lambda$

In this case, the long-time dynamics depend on the starting point $P$ of the shock. Using the notation introduced in Lemma 4, let $A_{-\infty}=\frac{1}{2 \sqrt{\lambda}} \log \left(\frac{\alpha}{2 \lambda}\right)$. Recall that for $\xi \in(-\infty, \Omega], A_{\xi}$ is location of the vertical asymptote for the characteristic curve $x_{\xi}(\tau)$.

Theorem 8 (Shock capture). If $\alpha<2 \lambda$ and $P \leq \Omega$, then the shock is blocked by the linear soliton (Figure 25), that is

$$
\begin{equation*}
\lim _{\tau \rightarrow \infty} s(\tau)=\Sigma \tag{9.31}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{-\infty} \leq \Sigma \leq A_{P} \leq 0 \tag{9.32}
\end{equation*}
$$

Proof. We have already seen that the shock increases either to infinity or to a finite supremum, and that the shock speed is the average value of the slopes of the left and right impinging characteristic curves. The right impinging curves have zero slope so contribute nothing to the shock speed. By Lemma 4, all characteristic curves originating from $(-\infty, P)$ approach vertical asymptotes in the interval $\left(A_{-\infty}, A_{P}\right)$. The left impinging slopes can therefore contribute nothing to the shock speed at or beyond $A_{p}$. This shows that the shock can not pass beyond $A_{P}$ because it can not have positive slope beyond $A_{p}$. To see that it can not stop before $A_{-\infty}$, pick a point $X<A_{-\infty}$.

Then $R_{0}^{2}(X)<R_{0}^{2}\left(A_{-\infty}\right)=\alpha$ so $\alpha-R_{0}^{2}(X)=\epsilon>0$ for some $\epsilon$. For all time, the slope of the shock curve at $X$ satisfies

$$
\left.\frac{d s}{d \tau}\right|_{s=x}=\frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}\right)-R_{0}^{2}(X)\right)>\epsilon
$$

Therefore the shock can not stop at any point $X<A_{-\infty}$.
Theorem 9. If $\alpha<2 \lambda$ and $P \leq \Omega$, the solution $v(x, \tau)$ of (6.20), (6.21) approaches the steady state solution $v_{\infty}(x)$ given by

$$
v_{\infty}(x)= \begin{cases}\alpha-R_{0}^{2}(x) & x \leq A_{-\infty}  \tag{9.33}\\ 0 & x>A_{-\infty}\end{cases}
$$

Figure 17 b shows the profile (9.33). It is has a discontinuous derivative at $x=A_{-\infty}$ but has no jump discontinuity. Theorem 9 thus shows that when the initial shock strength (the size of the jump) is smaller than the amplitude of the linear soliton, the shock strength is pushed to zero as time goes on. This is in contrast to the long-time behavior of $v(x, \tau)$ in the case $\alpha>2 \lambda$. Theorem 7 shows that in this case, the shock strength is weakened to some nonzero minimum, but then returns to its original value as $\tau$ increases to infinity.

Proof. Let $\xi_{X}(\tau)$, or just $\xi_{X}$, denote the left pullbacks of the vertical line $x=X$. $\xi_{X}$ must decrease either to minus infinity or to a finite infimum as $\tau \rightarrow \infty$. It is straightforward to show that if $X \leq A_{-\infty}$, then $\xi_{X}(\tau) \rightarrow-\infty$. If $A_{-\infty}<X \leq A_{P}$, then $X$ is the vertical asymptote location for some characteristic curve starting to the left of $X$. This follows from the definition of $A_{\xi}$. Suppose the curve starting from the point $Y$ has its vertical asymptote at $X$. It is then straightforward to show (using the definition of an infimum) that the left pullbacks of the vertical line $x=X$ decrease to the point $Y$ as $\tau \rightarrow \infty$. These observations imply that

$$
\lim _{\tau \rightarrow \infty} v(X, \tau)=\lim _{\tau \rightarrow \infty} \alpha+R_{0}^{2}\left(\xi_{X}\right)-R_{0}^{2}(X)=\alpha-R_{0}^{2}(X), \quad \text { if } X \leq A_{-\infty},
$$

and

$$
\lim _{\tau \rightarrow \infty} v(X, \tau)=\lim _{\tau \rightarrow \infty} \alpha+R_{0}^{2}\left(\xi_{X}\right)-R_{0}^{2}(X)=0, \quad \text { if } A_{-\infty}<X \leq A_{p}
$$

Since $v=0$ to the right of the shock, the theorem is proved.
Theorems 8 and 9 addressed the case $P \leq \Omega$ (and $\alpha<2 \lambda$ ). The next two theorems cover the case $P \geq 0$ (and $\alpha<2 \lambda$ ).

Theorem 10 (Shock escape with speed increase). If $\alpha<2 \lambda$ and $P \geq 0$, the shock passes through the linear soliton, and the shock speed converges to $\lambda$ (Figure 24).

Proof. Fix a point $X>P$. We show the shock speed can not go to zero at $X$, so the shock can not converge to a finite supremum. Since $X>0$ and $R_{0}^{2}$ is decreasing on $(0, \infty)$, then $2 \lambda-R_{0}^{2}(X)=\epsilon>0$ for some $\epsilon$. Since $P \geq 0$, characteristic curves starting to the left of $\Omega$ can not impinge upon the shock. This is because all of these curves hit vertical asymptotes before 0 . The left pullbacks $\xi_{s}^{-}$must therefore decrease to a finite infimum, say $k$, with $k \geq \Omega$. We will show that $k=\Omega$. But first note that at any time $\tau$, the slope of the shock curve at $X$ satisfies

$$
\begin{align*}
\left.\frac{d s}{d \tau}\right|_{s=X} & =\frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(X)\right)  \tag{9.34}\\
& >\frac{1}{2}\left(\alpha+R_{0}^{2}(\Omega)-R_{0}^{2}(X)\right) \\
& =\frac{1}{2}\left(2 \lambda-R_{0}^{2}(X)\right) \\
& >\epsilon, \tag{9.35}
\end{align*}
$$

so the shock must increase to infinity. To prove that its long-time speed is $\lambda$, we show that $k=\Omega$. Since the characteristic curve $x_{\Omega}(\tau)$ asymptotes at the line $x=0, x_{\Omega+\epsilon}(\tau)$ intersects the line $x=0$ at a time $\tau_{\epsilon}$ which goes to infinity as $\epsilon \rightarrow 0$. Therefore, characteristic curves starting close enough to $\Omega$ can be made to intersect the line $x=0$ arbitrarily high up. Curves starting close enough to $\Omega$ will thus pass over any transient shock that may have developed (see Theorem 5). By looking at the slopes of these characteristic curves, we can show that they must eventually intersect the shock. Therefore, the left pullbacks of the shock must converge to $\Omega$. That is,

$$
k=\lim _{\tau \rightarrow \infty} \xi_{s}^{-}(\tau)=\Omega
$$

As a result

$$
\lim _{\tau \rightarrow \infty} \frac{d s}{d \tau}=\lim _{\tau \rightarrow \infty} \frac{1}{2}\left(\alpha+R_{0}^{2}\left(\xi_{s}^{-}\right)-R_{0}^{2}(s)\right)=\frac{1}{2}\left(\alpha+R_{0}^{2}(\Omega)\right)=\lambda
$$

which concludes the proof.
Theorem 11. If $\alpha<2 \lambda$ and $P \geq 0$, then the long-time structure of the solution of equations (6.20), (6.21) is given by the formula

$$
v(x, \tau) \rightarrow \begin{cases}\alpha-R_{0}^{2}(x) & x \leq A_{-\infty}  \tag{9.36}\\ 0 & A_{-\infty}<x \leq 0 \\ 2 \lambda-R_{0}^{2}(x) & 0<x \leq s(\tau) \\ 0 & x>s(\tau)\end{cases}
$$

Figure 17c shows the profile (9.36). In the case $\alpha>2 \lambda$, we saw that when the shock escaped, its strength returned to its original state. In contrast to this, when $\alpha<2 \lambda$ and the shock escapes, its strength increases to $2 \lambda$, the amplitude of the linear soliton. In this sense, the effect of the dispersive wave on the shock wave is nonlocal. Though the shock passes through the linear soliton, it does not return to its original speed as it did in the case $\alpha>2 \lambda$; it undergoes a permanent strength increase.

Proof. The proof comes down to looking at the left pullbacks $\xi_{X}$ of the vertical line $x=X$ when $X$ is in each of the intervals on the right-hand side of equation (9.36). The arguments used in the proof of Theorem 9 show that for $X \leq A_{-\infty}, \xi x \rightarrow-\infty$, and for $A_{-\infty}<X \leq 0, \xi_{X} \rightarrow Y$ where $A_{Y}=X$. Therefore,

$$
\lim _{\tau \rightarrow \infty} v(X, \tau)=\lim _{\tau \rightarrow \infty} \alpha+R_{0}^{2}(\xi x)-R_{0}^{2}(X)= \begin{cases}\alpha-R_{0}^{2}(X) & X \leq A_{-\infty} \\ 0 & A_{-\infty}<X \leq 0\end{cases}
$$

For $0<X<s(\tau)$, the arguments used in the proof of Theorem 10 show that $\xi_{X}(\tau) \rightarrow$ $\Omega$. So for $0<X<s(\tau)$, we have

$$
\lim _{\tau \rightarrow \infty} v(X, \tau)=\lim _{\tau \rightarrow \infty} \alpha+R_{0}^{2}\left(\xi_{X}\right)-R_{0}^{2}(X)=\alpha+R_{0}^{2}(\Omega)-R_{0}^{2}(X)=2 \lambda-R_{0}^{2}(X)
$$

As always, $v=0$ to the right of the shock, hence the theorem is proved.


Figure 17: (a) Long-time profile (9.29) for $v$ in the case $\alpha>2 \lambda$. Shock strength decreases to a nonzero minimum, then returns to $\alpha$ as $\tau \rightarrow \infty$. (b) Long-time profile (9.33) in the case $\alpha<2 \lambda, P \leq \Omega$. Shock strength goes to zero as $\tau \rightarrow \infty$. This is a steady-state solution of equation (6.20) (c) Long-time profile (9.36) in the case $\alpha<2 \lambda, P \geq 0$. Shock strength increases from $\alpha$ to $2 \lambda$.

We next consider the case $\alpha<2 \lambda$ and $P \in(\Omega, 0)$. We make the following observations: If $s$ starts at $P$ and $\Omega<P<0$, the characteristic curve $x_{\Omega}(\tau)$ may impinge on the shock curve or may not. If it does, all of the other left impinging characteristic curves for later times originate from points left of $\Omega$. Thus if $x_{\Omega}(\tau)$ impinges on the shock curve, the shock will stop in the interval $\left[A_{-\infty}, 0\right]$ because the left impinging characteristics have asymptotes in this interval. Alternatively, if the shock increases to the point $s=0$ before $x_{\Omega}$ impinges upon it, none of the characteristic curves which asymptote in $\left[A_{-\infty}, 0\right]$ can reach the shock. In this case, the shock will escape. $\mathrm{Re}-$ gardless of $P$, the initial slope of the shock is $\alpha / 2$, while the initial slope of $x_{\Omega}(\tau)$ is $\alpha$. Thus one would expect $s$ to intersect $x_{\Omega}$ if it starts close to $\Omega$, or to reach 0 if it starts close enough to 0 . Numerical trials confirm this: they show that if $P$ is negative but near 0 , the shock escapes, and if $P$ is near but larger than $\Omega$, the shock stops before 0 . These observations lead to the following conjecture:

Conjecture Suppose $\alpha<2 \lambda$. There exists a point $\Lambda$ with $\Omega<\Lambda<0$ such that for $P<\Lambda$ the shock stops in the interval $\left[A_{-\infty}, 0\right]$ and has the long-time structure (9.33). For $P>\Lambda$, the shock escapes with long-time structure given by (9.36).

## 10 Numerical experiments

In this section we present numerical solutions ${ }^{1}$ of the leading order shock equations (6.18), (6.20) of Problems 1 and 2, respectively. In every case, the initial data is

$$
v(x, 0)= \begin{cases}1 & x \leq P  \tag{10.1}\\ 0 & x>P\end{cases}
$$

This corresponds to the choice $\alpha=1$ in equation (5.6) on page 48. In all of the figures, $v$ is compared to the solution of the conservation law

$$
\begin{equation*}
u_{\tau}+u u_{x}=0 \tag{10.2}
\end{equation*}
$$

[^1]with data (10.1). The solution of (10.2) in each case is shown in dashed lines. Both shock equations reduce to (10.2) when the dispersive wave amplitude $\lambda \rightarrow 0$.

The first three examples (Figures 18, 19, 20) are numerical solutions of equation (6.18), the effective shock equation of Problem 1. The evolution is described in Theorems 2 and 3. In each case, $\alpha=1$ and $\lambda=0.55$, so the values of $N_{1}$ and $N_{2}$ defined in Theorem 3 are approximately $\mp 1.28$, respectively. The only difference in each run is the value of the shock starting point $P$. In Figure 18, $P=0.5$, which is between $N_{1}$ and $N_{2}$, so the shock move backwards to the stopping point $N_{1}$. In Figure 19, $P=-2.5<N_{1}$, and so the shock moves forwards to the same stopping point. In Figure 20, $P=1.5>N_{2}$, and the shock travels right without stopping.

The next five figures (21-25) show numerical solutions of equation (6.20), the shock equation of Problem 2. Figures 21 and 22 show the transient shock phenomenon described in Theorem 5. In both runs, $\lambda=0.8$ but $P$ varies. In Figure 21, $P=6$, and the transient shock develops, while in Figure 22, $P=2.5$, and the transient shock is suppressed. Figure 23 shows an example of shock escape with no speed change as described in Theorems 6 and 7. Here, $\lambda=0.3$ and $P=-0.5$. In Figure 24, $\lambda=1.5$ and $P=0$, so Theorems 10 and 11 imply that the shock escapes but increases in size from $\alpha$ to $2 \lambda$. In agreement with formula (9.36), $v$ decreases to 0 in the interval $\left[A_{-\infty}, 0\right]$, where, in this case, $A_{-\infty} \approx-0.45$. The last example, Figure 25 , shows the shock capture described in Theorems 8 and 9. In this example, $\lambda=1.5$ and $P=-0.5$, which is to the left of $\Omega \approx-0.17$ (defined in equation (9.14)).

Two comparisons are worth emphasizing. First, in Figures 24 and 25, $\lambda$ is the same, only $P$ varies. Thus for a fixed shock strength of 1 and linear soliton amplitude of 3 , starting the shock to the left of the soliton results in shock capture, while starting it in the center of the soliton results in shock escape. Second, in Figures 23 and 25, $P$ is the same but $\lambda$ varies. This comparison shows that for a fixed shock strength of 1 and a fixed starting point of -0.5 , if the soliton's amplitude is small enough, the shock will pass through it (Figure 23), but if the amplitude is larger, the shock will be trapped (Figure 25).


Figure 18: Problem 1: Backwards shock capture. (a) $\tau=0$. (b) $\tau=3$. (c) $\tau=6$. (c) $\tau=9$. $\alpha=1, \lambda=0.55, P=0.5, N_{1} \approx-1.28$.


Figure 19: Problem 1: Forwards shock capture. (a) $\tau=0$. (b) $\tau=2$. (c) $\tau=4 . \alpha=1, \lambda=0.55, P=-2.5, N_{1} \approx$ -1.28 . (Dashed: $\lambda=0$ ).


Figure 19: Problem 1: Forwards shock capture. (d) $\tau=6$. (e) $\tau=8$. (f) $\tau=10 . \alpha=1, \lambda=0.55, P=-2.5, N_{1} \approx$ -1.28 . (Dashed: $\lambda=0$ ).


Figure 20: Problem 1: Shock escape. (a) $t=0$. (b) $t=2$. (c) $t=4$. $\alpha=1, \lambda=0.55, P=1.5, N_{1} \approx-1.28$. (Dashed: $\lambda=0$ ).


Figure 20: Problem 1: Shock escape. (d) $t=6$. (e) $t=8$. (f) $t=10 . \alpha=1, \lambda=0.55, P=1.5, N_{1} \approx-1.28$. (Dashed: $\lambda=0$ ).


Figure 21: Problem 2: Development of transient shock. (a) $t=0$. (b) $t=2$. (c) $t=4 . \alpha=1, \lambda=0.8, P=6$. (Dashed: $\lambda=0$ ).


Figure 21: Problem 2: Development of transient shock. (d) $t=7$. (e) $t=9$. (f) $t=10 . \alpha=1, \lambda=0.8, P=6$. (Dashed: $\lambda=0$ ).


Figure 22: Problem 2: Suppression of transient shock. (a) $t=0$. (b) $t=2$. (c) $t=4 . \alpha=1, \lambda=0.8, P=2.5$. (Dashed: $\lambda=0$ ).


Figure 22: Problem 2: Suppression of transient shock. (d) $t=7$. (e) $t=9$. (f) $t=10 . \alpha=1, \lambda=0.8, P=2.5$. (Dashed: $\lambda=0$ ).


Figure 23: Problem 2: Shock escape - no speed change. (a) $t=0$. (b) $t=2$. (c) $t=4$. $\alpha=1, \lambda=0.3, P=-0.5$. (Dashed: $\lambda=0$ ).


Figure 23: Problem 2: Shock escape - no speed change. (d) $t=6$. (e) $t=8$. (f) $t=10 . \alpha=1, \lambda=0.3, P=-0.5$. (Dashed: $\lambda=0$ ).


Figure 24: Problem 2: Shock escape with speed change. (a) $t=0$. (b) $t=0.02$. (c) $t=0.3 . \alpha=1, \lambda=1.5, P=0$, $A_{-\infty} \approx-0.45$. (Dashed: $\lambda=0$ ).


Figure 24: Problem 2: Shock escape with speed change. (d) $t=0.7$. (e) $t=1$. (f) $t=1.5 . \alpha=1, \lambda=1.5, P=0$, $A_{-\infty} \approx-0.45$. (Dashed: $\lambda=0$ ).


Figure 25: Problem 2: Shock capture. (a) $t=0$. (b) $t=0.5$. (c) $t=1 . \alpha=1, \lambda=1.5, P=-0.5, A_{-\infty} \approx-0.45$, $\Omega \approx-0.17$. (Dashed: $\lambda=0$ ).


Figure 25: Problem 2: Shock capture. (d) $t=3$. (e) $t=5$. (f) $t=7 . \alpha=1, \lambda=1.5, P=-0.5, A_{-\infty} \approx-0.45$, $\Omega \approx-0.17$. (Dashed: $\lambda=0$ ).

## Appendix

In Chapters 1 and 2, system ( 0.9 ) (page 3) was analyzed in the weak coupling limit as well as in the incompressible limit. This system is one of a family of equations which model shock wave-dispersive interactions. In this appendix, several other models are identified and briefly compared to (0.9). In some cases, the work in Chapters 1 and 2 carries over without much alteration. The general form of all the systems considered here is

$$
\begin{align*}
& \epsilon u_{t}+W(u, E) u_{x}=F(u, E)  \tag{A1}\\
& i E_{t}+E_{x x}=V(u, E) E,
\end{align*}
$$

where

$$
\begin{aligned}
& W=\text { wavespeed coefficient } \\
& F=\text { forcing term } \\
& V=\text { potential } .
\end{aligned}
$$

System (0.9) is obtained by setting

$$
\begin{aligned}
& W=u \\
& F=-u\left(\delta|E|^{2}\right)_{\mathbf{x}} \\
& V=u .
\end{aligned}
$$

Other systems result from different choices of $W, F$, and $V$. One is

$$
\begin{align*}
& \epsilon u_{t}+\left(u-\delta|E|^{2}\right) u_{x}=0 \\
& i E_{t}+E_{x x}+u E=0 . \tag{A2}
\end{align*}
$$

When $\delta=0$ (weak coupling), all of Chapter 1 (with a slight modification to account for the sign change in the potential) is applicable. The work of Chapter 2 also applies, but the positions of the soliton and cusped exponential are switched. More specifically, consider the $\epsilon=0$ problem (with $\delta=1$ ). It has solutions $u=|E|^{2}$ or $u=c$, which lead
to the NLS equation for $E$ in the latter case, and the linear dispersive equation

$$
i E_{t}+E_{x x}+c E=0
$$

in the former. We set $c=0$, though generally it may depend on $t$. After carrying out the multi-scale expansions for $u, R$, and $\theta$, the effective shock equations about each $\epsilon=0$ solution are

Problem 1 $\quad u_{0}=R_{0}^{2}+v(x, \tau):$

$$
\begin{align*}
& v_{\tau}+v\left(v+R_{0}^{2}\right)_{x}=0  \tag{A3}\\
& R_{0}^{2}=2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} x)
\end{align*}
$$

Problem 2 $u_{0}=v(x, \tau)$ :

$$
\begin{align*}
& v_{\tau}+\left(v-R_{0}^{2}\right) v_{x}=0,  \tag{A4}\\
& R_{0}^{2}=2 \lambda \exp (-2 \sqrt{\lambda}|x|) .
\end{align*}
$$

The soliton now appears in the variable friction equation (A3), while the cusped exponential appears in the variable wavespeed equation (A4). The theorems in Chapter 2 describing the shock dynamics for each problem still hold because they did not make use of the formulas for $R_{0}^{2}$, only their shapes. If one wanted to focus only on solitonshock interactions, system (0.9) would model soliton-shock interactions in the case of a variable wavespeed coupling, and system (A2) would govern these interactions in the case of a variable friction coupling.

An interesting variation of ( 0.9 ) is given by

$$
\begin{align*}
& \epsilon u_{t}+u\left(u+\delta|E|^{2}\right)_{x}=0  \tag{A5}\\
& i E_{t}+E_{x x}+\left(u+|E|^{2}\right) E=0,
\end{align*}
$$

where a nonlinearity appears in the potential of the $E$ equation. The weak coupling limit is more difficult, yet Chapter 1, excluding the small diffusion limits, carries over. In particular, the steady-state Riemann problem and the transient problem can be solved with little modification. The main difference is that all of the equations for
$R$ contain an additional $R^{2}$ term, so the solutions for $R$ are less explicit. In the incompressible limit, system (A5) is, to leading order, identical to system (0.9), though the higher order correction terms are different. The effective shock equations are the same, but the $\epsilon=0$ problems are switched. That is, when $\epsilon=0$, the solutions are still $u=-|E|^{2}$ and $u=0$, but these give LS and NLS, respectively, for $E$, instead of NLS and LS, respectively, in (0.9). The two problems are:

Problem 1 $u_{0}=-R_{0}^{2}+v(x, \tau):$

$$
\begin{aligned}
& v_{\tau}+v\left(v+R_{0}^{2}\right)_{x}=0, \\
& R_{0}^{2}=2 \lambda \exp (-2 \sqrt{\lambda}|x|) .
\end{aligned}
$$

Problem 2 $u_{0}=v(x, \tau):$

$$
\begin{aligned}
& v_{\tau}+\left(v-R_{0}^{2}\right) v_{x}=0, \\
& R_{0}^{2}=2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} x) .
\end{aligned}
$$

Again, the shock dynamics of Chapter 2 still apply. As was done in system (A2), the soliton can be made to appear in the variable friction equation by considering the system

$$
\begin{align*}
& \epsilon u_{t}+\left(u-\delta|E|^{2}\right) u_{x}=0  \tag{A6}\\
& i E_{t}+E_{x x}-\left(u-|E|^{2}\right) E=0 .
\end{align*}
$$

Systems (A5) and (A6) can be thought of as modeling two different types of solitonshock interactions. The nonlinear potential in these systems leads to different dynamics at higher orders, but this is still being investigated.

A model whose associated effective shock equations are very different than the ones above and in Chapter 2 is

$$
\begin{align*}
& \epsilon u_{t}+\left(u+\delta|E|^{2}\right) u_{x}=0  \tag{A7}\\
& i E_{t}+E_{x x}-u E=0 .
\end{align*}
$$

The $E$ equation agrees with that of ( 0.9 ), but now the $x$ derivative in the first equation has been moved. The weakly coupled system is identical to the one considered in Chapter 1, but the incompressible limit is very different. The two problems are:

Problem 1 $u_{0}=-R_{0}^{2}+v(x, \tau):$

$$
\begin{align*}
& v_{\tau}+\left(v+R_{0}^{2}\right) v_{x}=0  \tag{A8}\\
& R_{0}^{2}=2 \lambda \operatorname{sech}^{2}(\sqrt{\lambda} x) \tag{A9}
\end{align*}
$$

Problem 2 $u_{0}=v(x, \tau)$ :

$$
\begin{align*}
& v_{\tau}+v\left(v-R_{0}^{2}\right)_{x}=0  \tag{A10}\\
& R_{0}^{2}=2 \lambda \exp (-2 \sqrt{\lambda}|x|) \tag{A11}
\end{align*}
$$

Compare (A8) to (A4) and (A10) to (A3). The sign difference has a great effect on the characteristic equations and on the resulting shock dynamics. In particular, one can immediately see that in (A8) no shock blocking occurs because the $+R_{0}^{2}$ term can only increase the shock speed. The shock dynamics for (A10) are also different and are currently being studied. Variations of system (A7) are

$$
\begin{gather*}
\epsilon u_{t}+u\left(u-\delta|E|^{2}\right)_{x}=0  \tag{A12}\\
i E_{t}+E_{x x}+u E=0, \\
\epsilon u_{t}+\left(u+\delta|E|^{2}\right) u_{x}=0  \tag{A13}\\
i E_{t}+E_{x x}+\left(u+|E|^{2}\right) E=0,
\end{gather*}
$$

and

$$
\begin{align*}
& \epsilon u_{t}+u\left(u-\delta|E|^{2}\right)_{x}=0 \\
& i E_{t}+E_{x x}-\left(u-|E|^{2}\right) E=0 . \tag{A14}
\end{align*}
$$

Systems (A12)-(A14) are related to (A10) as systems (A2)-(A4) are related to (0.9).

## References

[1] P.D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, SIAM, 1972.
[2] G.B. Whitham, Linear and Nonlinear Waves, Wiley, 1974.
[3] B.L. Roždestvenskii, N.N. Janenko, Systems of quasilinear equations and their applications to gas dynamics, Translations of Mathematical Monographs, Vol. 55, AMS, 1980.
[4] J.M. Burgers, The Nonlinear Diffusion Equation: Asymptotic Solutions and Statistical Problems, Reidel, 1974.
[5] E. Hopf, 'The partial differential equation $u_{t}+u u_{x}=\mu u_{x x}$ ', Comm. Pure App. Math. 3, 201-230, 1950.
[6] C.M. Dafermos, 'Contemporary issues in the dynamic behavior of continuous media', LCDS Lecture Notes \#85-1, 1985.
[7] Tai-Ping Liu, 'Nonlinear resonance for quasilinear hyperbolic equation', J. Math. Phys. 28 (11), Nov., 1987.
[8] B. Hayes, 'Stability of solutions to a destabilized Hopf equation', Comm. Pure App. Math., Vol. XIVII, 1-10, 1995.
[9] C.M. Dafermos, 'Asymptotic behavior of solutions of hyperbolic balance laws', pp. 521-533 in: Bifurcation Phenomena in Mathematical Physics and Related Topics, C. Bardos and D. Bessis, eds., Reidel, Dordrecht, 1980.
[10] A.N. Lyberopoulos, 'Asymptotic oscillations of the solutions of scalar conservation laws with convexity under the action of a linear excitation', Quart. Appl. Math. 48, 755-765, 1991.
[11] P.D. Lax, C.D. Levermore, 'The small dispersion limit for the KdV equation I', Comm. Pure App. Math. 36, 253, 1983.
[12] P.D. Lax, C.D. Levermore, 'The small dispersion limit for the KdV equation II', Comm. Pure App. Math. 36, 571, 1983.
[13] P.D. Lax, C.D. Levermore, 'The small dispersion limit for the KdV equation III', Comm. Pure App. Math. 36, 809, 1983.
[14] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Muira, 'Method for solving the KdV equation', Phys. Rev. Lett. 19, 1095, 1967.
[15] P.D. Lax, 'Integrals of nonlinear equations of evolution and solitary waves', Comm. Pure App. Math., Vol. XXI, 467, 1968.
[16] V.I. Karpman, Nonlinear Waves in Dispersive Media, Pergamon Press, 1975.
[17] T.H. Havelock, The Propagation of Disturbances in Dispersive Media, Stechert-Hafner, 1964.
[18] J.L. Bona, V.A. Douglis, O.A. Karakahian, W.R. McKinney, 'Conservative highorder schemes for the generalized Korteweg-de Vries equation', Phil. Trans. R. Soc. Lond. A 351, 107-164, 1995.
[19] J.L. Bona, M.E. Schonbeck, 'Traveling wave solutions to the Korteweg-de Vries Burgers equation', Proc. A Roy. Soc. Edinburgh 101A, 207-226, 1985.
[20] D. Jacobs, W.R. McKinney, M. Shearer, 'Traveling wave solutions of the modified Korteweg-de Vries Burgers equation', J. Diff. Equa. 116, 448-467, 1995.
[21] V.D. Djordjevic, L.G. Redekopp, 'On two-dimensional packets of capillary-gravity waves', J. Fluid Mech 79, part 4, 703-714, 1977.
[22] N. Yajima, M. Okawa, 'Formulation and interaction of sonic-Langmuir solitons', Prog. Theor. Phys. 56, 1719, 1976.
[23] M.S. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, 'The inverse scattering transform - Fourier analysis for nonlinear problems', Studies in Appl. Math. 53, 249, 1974.
[24] V.E. Zakharov, 'Collapse of Langmuir waves', Sov. Phys. 17, 153, 1977.
[25] J. Gibbons, S.G. Thornhill, M.J. Wardrop, D. Ter Harr, 'On the theory of Langmuir solitons', J. Plasma Physics 17 (2), 153, 1977.
[26] D.G. Ebin, 'The motion of slightly compressible fluids viewed as a motion with strong constraining force', Ann. of Math. 105, 141-200, 1977.
[27] S. Klainerman, A. Majda, 'Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids', Comm. Pure App. Math., Vol. XXXIV, 481-524, 1981.
[28] Y.S. Kivshar, B.A. Malomed, 'Dynamics of solitons in nearly integrable systems', Rev. Mod. Phys. 61, No. 4, Oct. 1989.
[29] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, 1982.
[30] S.H. Schochet, M.I. Weinstein, 'The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence', Comm. Math. Phys. 106, 569, 1986.
[31] T. Ozawa, Y. Tsutsumi, 'The nonlinear Schrödinger limit and the initial layer of the Zakharov equations', Diff. and Int. Equa. 5 (4), 721, July 1992.
[32] P.K. Newton, 'Wave interactions in the singular Zakharov system', J. Math. Phys. 32 (2), 431, Feb., 1991.
[33] R.M. Axel, P.K. Newton, 'The interaction of shocks with dispersive waves: I. Weak coupling limit', Studies in Appl. Math. 96, 201-246, 1996.
[34] W. Munc, 'Acoustic monitoring of ocean gyres', J. Fluid Mech. 173 , 43-53, 1986.
[35] R.C. Shockley, J. Northrop, P.G. Hansen, 'SOFAR propagation paths from Australia to Bermuda: comparison of signal speed algorithms and experiments', J. Acoust. Soc. Am. 71, 51-60, 1982.
[36] D.G. Crighton, A.P. Dowling, J.E. Ffowcs Williams, M. Heckl, F.G. Leppington, Modern Methods in Analytical Acoustics, Lecture Notes, Springer-Verlag, 1992.
[37] R. Cook, 'Penetration of a sonic boom into water', J. Acous. Soc. Am. 47, No. 5, 1430, 1970.
[38] K. Sawyers, 'Underwater sound pressure from sonic booms', J. Acous. Soc. Am. 44, No. 2, 523, 1968.
[39] V. Sparrow, 'The effect of aircraft speed on the perstiation of sonic boom noise into a flat ocean', research notes, 1993.
[40] M. Hussey, Diagnostic Ultrasound: An Introduction to the Interactions Between Ultrasound and Biological Tissues, Wiley, 1975.
[41] L. Hutchins, S. Leeman, 'Tissue parameter measurement and imaging', in Acoustical Imaging, Vol. 11, Ed. J.P. Powers, Plenum Press, 1982.
[42] S. Leeman, 'Ultrasound pulse propagation in dispersive media', Phys. Med. Biol., 25:481, 1980.
[43] E.J. Smith, 'Observations of interplanetary shocks: recent progress', Space Sci. Review 34, 101, 1983.
[44] C.F. Kennel, F.L. Scarf, F.V. Coroniti, E.J. Smith, D.A. Gumett, 'Nonlocal plasma turbulence associated with interplanetary shocks', J. Geo. Res. 87, A1, 17, 1982.
[45] G.S. Bisnovatyi-Kogan, S.A. Silich, 'Shock-wave propagation in the nonuniform interstellar medium', Rev. Mod. Phys. 67, No. 3, 661, 1995.
[46] O. Oleinik, 'Discontinuous solutions of nonlinear differential equations', Usp. Mat. Nauk. 12, 3, 1957, English trans: Amer. Math. Soc. Transl. Ser. 2 26, 95, 1962.
[47] J.K. Hale, H. Koçak, Dynamics and Bifurcations, Springer-Verlag, 1991.
[48] P.D. Lax, 'Shock waves, increase of entropy and loss of information', in Seminar on Nonlinear PDE, Ed. S.S. Chern, MSRI Pub. 2, Springer 1984.
[49] M. Slemrod, 'Interrelationships among mechanics, numerical analysis, compensated compactness, and oscillation theory', in Oscillation Theory, Computation, and Methods of Compensated Compactness, Eds. Dafermos, Eriksen, Kinderlehrer, Slemrod, IMA Vol.2, Springer-Verlag, 1986.
[50] J. Smoller, Shock Waves and Reaction Diffiusion Equations, SpringerVerlag, 1983.
[51] J. Kevorkian, J. Cole, Perturbation Methods in Applied Mathematics, Applied Math. Sci. Vol. 34, Springer, 1980.
[52] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, 1978.
[53] J. Kevorkian, Partial Differential Equations: Analytical Solution Techniques, Wadsworth \& Brooks/Cole, 1990.
[54] M.V. Goldman, 'Strong turbulence of plasma waves', Rev. Mod. Phys. 56 (4), 709, 1984.

## Vita

Ralph Axel was born in Great Neck, New York in 1968. In 1990, he obtained a B.S. in mathematics from the University of Massachusetts. As a graduate student at the University of Illinois, he received an M.S. in 1994 and a Ph.D. in 1996, both in mathematics. For portions of his graduate career, he was awarded a Department of Education National Needs Fellowship. As a teaching assistant, he was a three time member of the university wide Incomplete List of Excellent Teachers. He is a coauthor of the article The interaction of shocks with dispersive waves: I. Weak coupling limit, which appears in Studies in Applied Mathematics, 1996. His interests include literature, music and mountain biking. He is currently pursuing a career in finance.


[^0]:    ${ }^{1}$ This work appears in [33].

[^1]:    ${ }^{1}$ The software was written by Peiji Chen, Department of Aerospace Engineering, University of Southern Californis, Los Angeles, CA.

