# N-VORTEX PROBLEM ON A ROTATING SPHERE 

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## Dedication

I dedicate this dissertation to my parents, Ahmad and Mehrnaz, for their unconditional love, support and encouragement throughout my life.

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#### Abstract

The evolution, interaction, and scattering of 2 N -point vortices grouped into equal and opposite pairs ( N -dipoles) on a rotating unit sphere is studied. A new coordinate system made up of the centers-of-vorticity and centroids associated with each dipole is introduced. With these coordinates, the nonlinear equations for an isolated dipole diagonalize and one directly obtains the equation for geodesic motion on the sphere for the dipole centroid. When two or more dipoles interact, the equations are viewed as an interacting billiard system on the sphere - charged billiards - with long range interactions causing the centroid trajectories to deviate from their geodesic paths. Canonical interactions are studied both with and without rotation. For two dipoles, the four basic interactions are described as exchange scattering, non-exchange scattering, loop scattering (head-on) and loop scattering (chasing) interactions. For three or more dipoles, one obtains a richer variety of interactions, although the interactions identified in the two-dipole case remain fundamental.


Keywords: N-vortex problem; Dipole scattering; Charged billiard equations.

## Chapter 1

## Introduction to $\mathbf{N}$-vortex problem

In the last 20 years, there has been a resurgence of interest in the N -vortex problem. Motivated mostly by advances in dynamical systems and computational techniques, the focus has by and large been on the dynamics of point vortices in the plane, both integrable and non-integrable configurations. From a geophysical point of view, however, it is also important to study the motion of vortices on a sphere if one is interested in questions related to large scale motions and mixing in the atmosphere and oceans. When the scale of motion is on the order of the radius of the Earth, the topology of the sphere becomes important and one can no longer use a tangent plane approximation. In addition to curvature effects, rotational effects can also play an important role, particularly when one wants to track regions of concentrated vorticity (such as cyclones and hurricanes) on timescales comparable to or longer than a day. In general, of course, both the curvature of the sphere as well as its rotation will influence the dynamics. In this effort we bring the effects of the rotation into account and thus study both how geometry influences the vortex motion and how the rotation of will effect the path of the motion. Papers that address the effects of rotation on the full sphere include Bogomolov (1985), DiBattista \& Polvani (1992), S.Friedlander (1975) Newton \& Shokraneh (2004). This effect, of course, introduces significant complications to the dynamics and, generally speaking, there has not been much analytical progress on this problem.

There is, by now, a substantial and growing literature devoted to the N -vortex problem on the non-rotating sphere $(\Omega=0)$ and much is known regarding integrability (see Borisov \& Pavlov (1998), Borisov \& Lebedev (1998), Kidambi \& Newton (1998), Kidambi \& Newton (2000)) non-integrability (see Bagrets \& Bagrets (1997),Newton \& Ross (2005)) collisions (see Kidambi \& Newton (1998), Kidambi \& Newton (1999)), fixed and relative equilibria (see Aref et al. (2003), Kidambi \& Newton (1998), Laurent-Polz (2002), Lim et al. (2001), Jamaloodeen \& Newton (2005)), and stability (see Polvani and Dritschel (1993), Cabral et
al. (2003), Kurakin (2004), Laurent-Polz et al. (2004), Pekarsky \& Marsden (1998)). MonteCarlo methods have been developed which identify extremal states (see Lim et al. (2003a,b)) and special numerical techniques that retain accuracy on theoretically conserved quantities are being developed (Marsden et al. (1999), Newton \& Khushalani (2002), Pullin \& Saffman (1991), Rowley \& Marsden (2000), Rowley et al. (2004), Zhang \& Qin (1993)). An overview of many of these topics can be found in Newton (2001), while a recent comprehensive survey of equilibrium configurations can be found in Aref et al. (2003). What motivates most of these efforts are applications to atmospheric fows, traditionally treated under the $\beta$-plane approximation (see Gill (1982)). While $\beta$-plane models include Coriolis effects, they are local and only remain valid in a restricted latitudinal strip about which the tangent plane approximation is invoked. Thus, when one is interested in tracking vorticity over long distances, or when global velocity fields and streamline patterns are of interest (Kidambi \& Newton (2000)) typically a full spherical treatment is required. The problem has spawned several Ph.D. theses, both from the geophysical fluid dynamics perspective (see Chern (1991), Nevin (1993), and DiBattista (1997)) as well as the nonlinear dynamics point of view (see Kidambi (1999), Jamaloodeen (2000), Laurent-Polz (2002), Nebus (2003), Khushalani (2004)). The papers of Bogomolov (1985), DiBattista \& Polvani (1998), Klyatskin \& Resnik (1989) treat the fully coupled 'barotropic' model on the sphere where the vortices influence the background rotation and in turn, the evolving background field influences the vortices. This two-way coupling allows for the generation of RossbyHaurwitz waves on the sphere which are known, for example, to trigger instabilities in the vortex configuration. However, because the background vorticity is not localized, Bogomolov's (1985) equations are integro-differential equations which typically must be treated numerically. Likewise, Klyatskin \& Resnik (1989) resort to using use a short time approximation (Taylor expansion) to show that an isolated point vortex, coupled to the background field, moves along a northwesterly curved trajectory on the sphere, in qualitative agreement with what is known about the trajectories of hurricane paths in the northern hemisphere. The numerical study in DiBattista \& Polvani (1998) treats the interaction of a vortex dipole with a background distribution in the form of constant vorticity strips on the sphere which initially model solid body rotation, while Polvani \& Dritschel (1993) treat both wave and vortex dynamics on the sphere using contour dynamics techniques. In the simpler model which will be studied in this thesis, since the vortex motion does not affect the background velocity field which remains in solid body form, the system retains its finite-
dimensional structure, much like the models that focus on the non-rotating sphere and thus can be treated analytically. This model can be considered a limiting case of the two-way coupled model in the limit in which the background field is strong compared to the strength of the embedded vortices. The price we pay is that this one-way coupled system is not capable of generating Rossby-Haurwitz waves.

In order to narrowly focus on a subject, I started with Kidambi \& Newton (2000) and the streamline topologies for integrable vortex motion on a sphere. They have described the instantaneous streamline patterns that occur on the surface of a two-dimensional sphere in the presence of point vortices of general strength. They have categorized all possible instantaneous streamline patterns and described their stagnation point structure for the cases of two and three vortices. It is found that for the case of two vortices, the only nondegenerate topologies that arise are a figure eight (lemniscate) or a limacon, which are homotopically equivalent.


Figure.1. Continous deformation of a lemniscate to a limacon. In four steps (a)-(d), the sphere is pushed through the left top.

For the case of three vortices, there are 12 topologically distinct primitives, from which an additional 23 patterns can be produced via continuous deformations on the sphere
(homotopies). All possible streamline patterns that arise from three vortex motion can be obtained via linear superposition of the primitive topologies and their homotopic equivalents. In this sense, the primitives can be viewed as the 'building blocks' for the general flow patterns. The classification into 12 primitive topologies is shown in fig.2(a). the left column lists the number of saddle points occurring in the figure. The numbers along the bottom of each figure indicate the number of homoclinic-hetroclinic - trihedroclinic loops in each figure. Hence, the upper left figure is the 'least complex', while that on the bottom right is the 'most complex'.


Figure.2(a). Three vortex primitive chart showing the 12 primitive topologies. Number down left denotes the number of saddle points. The three numbers under each figure refer to the number of homoclinic-heteroclinictriheteroclinic loops.

It is important to understand that each primitive can be continuously deformed on the surface of the sphere to a visually distinct but topologically equivalent pattern as was shown, for example, in Fig.1. Hence, each of the 12 figures represents a homotopy equivalence class.

They have shown as in Fig.2(b). the primitives and their topologically equivalent figures, of which there are 23. They have mentioned that any given streamline configuration associated with the three vortex problem will be made up of a general combination of the primitives and their topological equivalents. Such streamline patterns are shown in Fig.3. which is a combination of lemniscate and limacon.


Figure.2(b). Twenty three homotopic equivalent figures obtained by continuously deforming each of the 12 primitives shown in Fig. 2(a).


Figure.3. Typical three vortex streamline pattern. Shown is a combination of a lemniscate and limacon formed by a three vortex cluster: (a) front of sphere; (b) back of sphere; (c) stereographic projection

The question remains as to how the streamline topologies evolve dynamically? In particular, can one identify the bifurcations that take place from one pattern to another as the vortices evolve? That is what I am going to study in this thesis and my ultimate goal would be to compare my results for simulated streamline evolution on the sphere with real data I have collected from NOAA website. (Appendix I.)

In order to study the bifurcation of topologies, we need first to know how the point vortices dynamically evolve on the sphere. High accuracy in the computation is needed so we typically use a variable time-step 7th-8th order Runge-Kutta solver. The dynamics, of course, may result in chaotic motions which we study by starting with simplest case of dipoles and then expand to more complex cases.

### 1.1 Cartesian coordinate formulation in the plane

We briefly review the main equations we treat in this study. The system of N-point vortices in the plane can be expressed conveniently as

$$
\begin{gather*}
\dot{X}_{\alpha}=\frac{-1}{2 \Pi}\left(\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\alpha}\left(Y_{\alpha}-Y_{\beta}\right)}{l^{2}{ }_{\alpha \beta}}\right)  \tag{1.1}\\
\dot{Y}_{\alpha}=\frac{-1}{2 \Pi}\left(\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\alpha}\left(X_{\alpha}-X_{\beta}\right)}{l^{2}{ }_{\alpha \beta}}\right) \tag{1.2}
\end{gather*}
$$

Where $l_{\alpha \beta}=\left\|X_{\alpha}-X_{\beta}\right\|$ are the intervortical distances. The system is more compactly expressed in complex notation $\mathrm{Z}_{\alpha}(\mathrm{t})=\mathrm{X}_{\alpha}(\mathrm{t})+\mathrm{i} \mathrm{Y}_{\alpha}(\mathrm{t})$,

$$
\begin{equation*}
\dot{Z}_{\alpha}=\frac{i}{2 \Pi}\left(\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\beta}\left(Z_{\alpha}-Z_{\beta}\right)}{\left|Z_{\alpha}-Z_{\beta}\right|^{2}}\right) \tag{1.3}
\end{equation*}
$$

### 1.2 Vortex dynamics on sphere

$$
\begin{gather*}
\dot{\theta}_{\alpha}=\frac{-1}{2 \Pi}\left(\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\beta} \sin \left(\theta_{\beta}\right) \sin \left(\phi_{\alpha}-\phi_{\beta}\right)}{l^{2}{ }_{\alpha \beta}}\right)  \tag{1.4}\\
\sin \left(\theta_{\alpha}\right) \dot{\phi}_{\alpha}=\frac{1}{2 \Pi} \sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\beta} \gamma_{\alpha \beta}}{l^{2}{ }_{\alpha \beta}} \tag{1.5}
\end{gather*}
$$

Using standard spherical coordinates, and where

$$
\gamma_{\alpha \beta}=\sin \left(\theta_{\alpha}\right) \cos \left(\theta_{\beta}\right)-\cos \left(\theta_{\alpha}\right) \sin \left(\theta_{\beta}\right) \cos \left(\phi_{\alpha}-\phi_{\beta}\right)
$$

### 1.3 Unifying planar and spherical formulas

We can unify the planar and spherical formulas by writing the velocity field at an arbitrary point X due to point vortex of strength $\Gamma_{\alpha}$ located at position $\vec{X}_{\alpha}$ as

$$
\begin{equation*}
\dot{\vec{X}}=\frac{\Gamma_{\alpha}}{2 \Pi} \cdot \frac{\vec{n}_{\alpha} \times\left(\vec{X}-\vec{X}_{\alpha}\right)}{\left\|\vec{X}-\vec{X}_{\alpha}\right\|^{2}} \tag{1.6}
\end{equation*}
$$

Where $\mathrm{n}_{\alpha}$ is the unit normal vector to the surface at the vortex location $\vec{X}_{\alpha}$. The velocity due to a collection of N such vortices is then

$$
\begin{equation*}
\dot{\vec{X}}=\sum_{\alpha=1}^{N} \frac{\Gamma_{\alpha}}{2 \Pi} \cdot \frac{\vec{n}_{\alpha} \times\left(\vec{X}-\vec{X}_{\alpha}\right)}{\left\|\vec{X}-\vec{X}_{\alpha}\right\|^{2}} \tag{1.7}
\end{equation*}
$$

Which gives the equations for a collection of N -vortices

$$
\begin{equation*}
\stackrel{\vec{X}}{\beta}=\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\alpha}}{2 \Pi} \cdot \frac{\vec{n}_{\alpha} \times\left(\vec{X}_{\beta}-\vec{X}_{\alpha}\right)}{\left\|\vec{X}_{\beta}-\vec{X}_{\alpha}\right\|^{2}} \tag{1.8}
\end{equation*}
$$

Where $\beta=1, \ldots . . \mathrm{N}$. The characteristic feature of the surface enters only through specifying the normal vector at each vortex location. For planar problem $\mathrm{n}_{\alpha}=\mathrm{e}_{\mathrm{z}}$, and the equation become

$$
\begin{equation*}
\stackrel{\vec{X}}{\beta}=\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\alpha}}{2 \Pi} \cdot \frac{\vec{e}_{\alpha} \times\left(\vec{X}_{\beta}-\vec{X}_{\alpha}\right)}{\left\|\vec{X}_{\beta}-\vec{X}_{\alpha}\right\|^{2}} \tag{1.9}
\end{equation*}
$$

On the sphere with radius $\mathrm{R}, \mathrm{n}_{\alpha}=\vec{X}_{\alpha} / \mathrm{R}$ and the equations are

$$
\begin{equation*}
\stackrel{\vec{X}}{\beta}=\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\alpha}}{2 \Pi} \cdot \frac{\left(\frac{\vec{X}_{\alpha}}{R}\right) \times\left(\vec{X}_{\beta}-\vec{X}_{\alpha}\right)}{\left\|\vec{X}_{\beta}-\vec{X}_{\alpha}\right\|^{2}} \tag{1.10}
\end{equation*}
$$

One, of course, has the additional constraint on the sphere that $\left\|\vec{X}_{\alpha}\right\|=\mathrm{R}$, Hence we can modify above equation as follow:

$$
\begin{equation*}
\stackrel{\rightharpoonup}{X}_{\beta}=\sum_{\beta \neq \alpha}^{N} \frac{\Gamma_{\alpha}}{2 \Pi} \cdot \frac{\vec{X}_{\alpha} \times \vec{X}_{\beta}-\vec{X}_{\alpha} \times \vec{X}_{\alpha}}{l^{2}{ }_{\alpha \beta} \cdot R} \tag{1.11}
\end{equation*}
$$

Now for substituting $l_{\alpha \beta}{ }^{2}$ if we consider a vector $\left\langle\vec{X}_{\alpha}-\vec{X}_{\beta}\right\rangle$ and multiply it by itself we will have:

$$
\begin{gather*}
l^{2}{ }_{\alpha \beta}=\left\langle\vec{X}_{\alpha}-\vec{X}_{\beta}\right\rangle \times\left\langle\vec{X}_{\alpha}-\vec{X}_{\beta}\right\rangle \\
l^{2}{ }_{\alpha \beta}=\left\|\vec{X}_{\alpha}\right\|^{2}+\left\|\vec{X}_{\beta}\right\|^{2}-2 \vec{X}_{\alpha} \cdot \vec{X}_{\beta}  \tag{1.12}\\
l^{2}{ }_{\alpha \beta}=2\left(R^{2}-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}\right)
\end{gather*}
$$

On the other hand we know that $\vec{X}_{\alpha} \times \vec{X}_{\alpha}=0$,it follows that:

$$
\begin{equation*}
\stackrel{\vec{X}}{\beta}=\frac{1}{4 \Pi R} \sum_{\beta \neq \alpha}^{N} \Gamma_{\alpha} \cdot \frac{\vec{X}_{\alpha} \times \vec{X}_{\beta}}{R^{2}-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}} \tag{1.13}
\end{equation*}
$$

In many respects, this is the most convenient way of writing the system on a sphere with no rotation.

### 1.4 Summary of Ph.D thesis

The thesis will be organized into four chapters and two appendices. Chapter 1 introduces the history and background associated with the problem. Chapter 2 describes the effects of alignment and rotation on the problem of N -vortices on a unit sphere and introduces simple linear transformations which help to decompose it into simpler interactions. Chapter 3 describes how we can rewrite our formula to be suitable for simulations via a high order Runga-Kutta method. Chapter 4 explores the fundamental interactions of two and three dipoles and then a general case of larger number of dipoles with different strengths. The two appendices contain preliminary work on streamline patterns which occour both in real data and in our models. Details follow below.

## 2- Equilibria: relative equilibria, One-frequency and multi frequency

The problem of N-point vortices moving on a rotating unit sphere is considered. Through a sequence of linear coordinate transformations which takes into account the orientation of the center of vorticity vector with respect to the axis of rotation, we show how to reduce the problem to that on a non-rotating sphere, where the center of vorticity vector is aligned with the $z$-axis. As a consequence, we prove that integrability on the rotating sphere is the same as on the non-rotating sphere, namely, the three-vortex problem on the rotating sphere is integrable for all vortex strengths, while the four-vortex problem is integrable in the special case where the center of vorticity is zero. Rigid multi-frequency configurations that retain their shape while rotating about two independent axes with two independent frequencies are obtained, and necessary conditions for one-frequency and two-frequency motion are derived. Examples including dipoles which exhibit global 'wobbling' and 'tumbling' dynamics, rings, and Platonic solid configurations are shown to undergo either periodic or quasiperiodic evolution on the rotating sphere depending on the ratio of the solid body rotational frequency $\Omega$ to the rotational frequency $\omega$ associated with the rigid structure.

## 3- Numerically solving the $\mathbf{N}$-vortex problem on a sphere

In chapter 1, formulas for calculating the effects of one vortex on another and the velocity vector for each vortex have been studied thoroughly. In this chapter, we will expand these formulas for the case of N -vortices and describe an accurate numerical solution, specifically for the case of N interacting dipoles.

## 4- Fundamental interactions of $\mathbf{N}$-vortex on a unit sphere

In this final chapter, we will study the fundamental interactions which occur between two and three dipoles. These interactions turn out to occur in more complicated settings, and allow us to interpret more general N-dipole interactions, between dipoles of different strengths.

## Chapter 2

## Equilibria: relative equilibria, One-frequency and multifrequency

In this chapter, the N -vortex problem on a rotating unit sphere is considered. It is convenient to formulate the problem in Cartesian coordinates, where the vector $\vec{X}_{\alpha} \in \mathfrak{R}^{3}$ points from the center of the unit sphere to the point vortex with strength $\Gamma_{\alpha}$ lying in the surface of the sphere, as shown in figure .4. Each point vortex moves under the collective influence of all the others and rotation is introduced by adding a solid body rotational component to the velocity field. The dynamical system we consider is given by

$$
\begin{gather*}
\stackrel{\rightharpoonup}{X}_{\alpha}=\frac{1}{4 \Pi} \sum_{\beta=1}^{N}{ }^{\prime} \Gamma_{\beta} \cdot \frac{\vec{X}_{\beta} \times \vec{X}_{\alpha}}{1-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}}+\Omega \hat{e}_{z} \times \vec{X}_{\alpha} \quad(\alpha=1, \ldots, N)  \tag{2.1}\\
\vec{X}_{\alpha} \in \mathfrak{R}^{3},\left\|\vec{X}_{\alpha}\right\|=1
\end{gather*}
$$

The prime on the summation reminds us that the singular term $\beta=\alpha$ is omitted and initially, the vortices are located at the given positions $\vec{X}_{\alpha}(0) \in \mathfrak{R}^{3},(\alpha=1, \ldots, N)$. The denominator $\operatorname{in}(2.1)$ is the chord distance between vortex $\Gamma_{\alpha}$ and $\Gamma_{\beta}$ since $\left\|\vec{X}_{\alpha}-\vec{X}_{\beta}\right\|^{2}=2\left(1-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}\right)$. In what follows, the center of vorticity vector J (also known as the momentum map) defined as

$$
\begin{equation*}
\vec{J}=\left(\sum_{\alpha=1}^{N} \Gamma_{\alpha} X_{\alpha}, \sum_{\alpha=1}^{N} \Gamma_{\alpha} Y_{\alpha}, \sum_{\alpha=1}^{N} \Gamma_{\alpha} Z_{\alpha}\right)=\left(J_{x}, J_{y}, J_{z}\right) \tag{2.2}
\end{equation*}
$$

plays a central role in our discussion.


Figure.4. N -vortex diagram on a unit rotating sphere, J rotates around $\hat{e}_{z}$ with frequency $\Omega$

The goal in this chapter is to make a simple observation that seems to have been missed previously, namely, that the mis-alignment of the center of vorticity vector with the axis of rotation is an important ingredient in understanding the dynamics of the vortices, and on its own can account for features such as the 'wobbling' and 'tumbling' modes seen previously in $\beta$-plane models. The key to understanding the ramifications of the mis-alignment is to understand how a certain time-dependent unitary operator, $\mathfrak{I}_{\Omega}{ }^{J}(t)$, affects trajectories on the aligned non-rotating sphere. It is this feature that we will explore in this chapter.

In §2-1 we transform system (2.1) to the corresponding equations for N -vortices on a nonrotating sphere. In §2-2 we align the $\mathbf{J}$ vector with the $z$-axis through two sequential linear transformations. We show that the general solution to (2.1) can be related, via a sequence of three linear mappings which define $\mathfrak{I}_{\Omega}{ }^{J}(t)$, to solutions on the non-rotating sphere where $\mathbf{J}$ is aligned with the z -axis. From this, we can conclude that, like the non-rotating sphere (Kidambi \& Newton (1998)), the 3- vortex problem on the rotating sphere is integrable for all vortex strengths. This is described in §2-3. §2-4 focuses on the conditions necessary for the existence of rigidly rotating configurations which maintain the mutual distances between each pair of vortices on the rotating sphere. These solutions contain two inherent frequencies, $(\Omega, \omega)$ and hence represent either periodic orbits ( $\Omega / \omega$ rational) or quasiperiodic orbits ( $\Omega / \omega$ irrational) of the original system (2.1). We then describe the evolution of dipoles, rings, and Platonic solid configuration on the rotating sphere.

## a) Solid-body rotation

Consider first just the solid-body term in (2.1),

$$
\begin{gather*}
\overrightarrow{\vec{X}}=\Omega \hat{e}_{z} \times \vec{X}=\left(-\Omega_{y}, \Omega_{x}, 0\right)  \tag{2.3}\\
\vec{X}_{\alpha} \in \mathfrak{R}^{3},\left\|\vec{X}_{\alpha}\right\|=1
\end{gather*}
$$

This can be solved by transforming the coordinates to a rotating reference frame via the linear transformation $X \in \mathfrak{R}^{3} \mapsto w \in \mathfrak{R}^{3}$, where

$$
\begin{equation*}
\vec{X}=M_{\Omega} \vec{w} \tag{2.4}
\end{equation*}
$$

$\operatorname{and}_{M_{\Omega}}$ is the rotation matrix about the z -axis

$$
M_{\Omega}=\left(\begin{array}{ccc}
\cos \Omega t & -\sin \Omega t & 0  \tag{2.5}\\
\sin \Omega t & \cos \Omega t & 0 \\
0 & 0 & 1
\end{array}\right)
$$

For future reference we note that $M_{\Omega}(0)=I$ and $M_{\Omega}$ is unitary matrix, hence has the property

$$
\begin{equation*}
M_{\Omega}^{T}=M_{\Omega}^{-1}, \tag{2.6}
\end{equation*}
$$

Inserting this into (2.3) yields

$$
\begin{equation*}
\dot{\vec{X}}=\dot{M}_{\Omega} \vec{w}+M_{\Omega} \dot{\vec{w}}=\Omega \hat{e}_{z} \times\left(M_{\Omega} \vec{w}\right) \tag{2.7}
\end{equation*}
$$

A straight forward calculation shows that

$$
\begin{equation*}
\dot{M}_{\Omega} \vec{w}=\Omega \hat{e}_{z} \times\left(M_{\Omega} \vec{w}\right)=\Omega \hat{e}_{z} \times \vec{X} . \tag{2.8}
\end{equation*}
$$

Thus (2.7) reduces to

$$
\begin{equation*}
M_{\Omega} \dot{w}=0 \Rightarrow w=w(0)=X(0), \tag{2.9}
\end{equation*}
$$

and the solution to (2.3) then becomes

$$
\begin{equation*}
\vec{X}(t)=M_{\Omega} \vec{X}(0) \tag{2.10}
\end{equation*}
$$

## b) The center of vorticity vector

Central to our approach is the center of vorticity vector (2.2), in particular it's orientation with respect to the axis of rotation. We first consider it's evolution by multiplying (2.1) by $\Gamma_{\alpha}$ and summing over $\alpha$

$$
\begin{equation*}
\sum_{\alpha=1}^{N} \Gamma_{\alpha} \dot{X}_{\alpha}=\frac{1}{4 \Pi} \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N}{ }^{\prime} \Gamma_{\alpha} \Gamma_{\beta} \cdot \frac{\vec{X}_{\beta} \times \vec{X}_{\alpha}}{1^{2}-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}}+\sum_{\alpha=1}^{N} \Omega \hat{e}_{z} \times \Gamma_{\alpha} \vec{X}_{\alpha} . \tag{2.11}
\end{equation*}
$$

In the right hand side of this equation, the first term is zero, hence what remains is

$$
\begin{equation*}
\sum_{\alpha=1}^{N} \Gamma_{\alpha} \dot{\vec{X}}_{\alpha}=\Omega \hat{e}_{z} \times \sum_{\alpha=1}^{N} \Gamma_{\alpha} \vec{X}_{\alpha} \tag{2.12}
\end{equation*}
$$

Thus $J$ satisfies

$$
\begin{equation*}
\dot{\vec{J}}=\Omega \hat{e}_{z} \times \vec{J} \tag{2.13}
\end{equation*}
$$

the same equation as (2.3). Hence, as in (2.10)

$$
\begin{equation*}
\vec{J}(t)=M_{\Omega} \times \vec{J}(0), \tag{2.14}
\end{equation*}
$$

from which we conclude that its length is constant since

$$
\begin{equation*}
\|\vec{J}\|^{2}=\langle\vec{J}, \vec{J}\rangle=\left\langle M_{\Omega} \vec{J}(0), M_{\Omega} \vec{J}(0)\right\rangle=\left\langle M_{\Omega}^{T} M_{\Omega} \vec{J}(0), \vec{J}(0)\right\rangle=\|\vec{J}(0)\|^{2} \tag{2.15}
\end{equation*}
$$

The components satisfy

$$
\begin{gather*}
J_{x}^{2}+J_{y}^{2}=C_{1}=\text { const } .  \tag{2.16}\\
J_{z}=C_{2}=\text { const } . \tag{2.17}
\end{gather*}
$$

A general configuration is depicted in figure 4. As the N -vortices evolve under their mutual interaction, the J vector rotates with frequency $\Omega$ about the z -axis, maintaining a fixed angle $\gamma$ with respect to the axis.


Figure.5. Alignment of J with $\hat{e}_{z}$ axis is obtained via a rotation through angle $\gamma_{\mathrm{z}}$ about the z -axis followed by a rotation $\gamma_{y}$ about y-axis

### 2.1 Transformation to a non-rotating sphere

To treat the full system (2.1), we first move to a rotating reference frame to absorb the solid body rotational term, Hence substitute (2.4) into (2.1), noting that

$$
\begin{equation*}
\vec{X}_{\alpha} \cdot \vec{X}_{\beta}=\left\langle\vec{X}_{\alpha}, \vec{X}_{\beta}\right\rangle=\left\langle M_{\Omega} \vec{w}_{\alpha}, M_{\Omega} \vec{w}_{\beta}\right\rangle \tag{2.18}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\vec{X}_{\alpha} \cdot \vec{X}_{\beta}=\left\langle M_{\Omega}^{T} M_{\Omega} \vec{w}_{\alpha}, \vec{w}_{\beta}\right\rangle=\vec{w}_{\alpha} \cdot \vec{w}_{\beta}, \tag{2.19}
\end{equation*}
$$

and as $M_{\Omega}$ is a pure rotation matrix, magnitude and the angle between $\vec{X}_{\alpha}$ and $\vec{X}_{\beta}$ remains constant therefore:

$$
\begin{equation*}
\vec{X}_{\alpha} \times \vec{X}_{\beta}=\left(M_{\Omega} \vec{w}_{\alpha}\right) \times\left(M_{\Omega} \vec{w}_{\beta}\right)=M_{\Omega}\left(\vec{w}_{\alpha} \times \vec{w}_{\beta}\right), \tag{2.20}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\dot{\vec{w}}_{\alpha}=\frac{1}{4 \Pi} \sum_{\beta=1}^{N}{ }^{\prime} \Gamma_{\beta} \frac{\vec{w}_{\beta} \times \vec{w}_{\alpha}}{\left(1-\vec{w}_{\alpha} \cdot \vec{w}_{\beta}\right)}, \quad \vec{w}_{\alpha}(0)=\vec{X}_{\alpha}(0), \quad(\alpha=1, \ldots, N) \tag{2.21}
\end{equation*}
$$

Hence transformation (2.4) takes solutions on the rotating sphere to solutions on the nonrotating sphere. The center of vorticity vector transforms as

$$
\begin{equation*}
\vec{J}=\sum_{\alpha=1}^{N} \Gamma_{\alpha} \vec{X}_{\alpha}=\sum_{\alpha=1}^{N} \Gamma_{\alpha} M_{\Omega} \vec{w}_{\alpha}=M_{\Omega} \sum_{\alpha=1}^{N} \Gamma_{\alpha} \vec{w}_{\alpha}=M_{\Omega} \hat{J} \tag{2.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{J}=\sum_{\alpha=1}^{N} \Gamma_{\alpha} w_{\alpha}=\left(\hat{J}_{x}, \hat{J}_{y}, \hat{J}_{z}\right)=J(0)=\text { const } . \tag{2.23}
\end{equation*}
$$

The initial configuration $\vec{w}_{\alpha}(0)=\vec{X}_{\alpha}(0),(\alpha=1, \ldots, N)$ defines the constant vector $\vec{J}$.

### 2.2 Alignment

We now rotate $\hat{J}$ so that it is aligned with the z - axis as shown in figure.5. First, we multiply by the matrix $M_{z}$ which rotates $\hat{J}$ around the z -axis so that it lies in the $(x, z)$ plane, then we multiply by $M_{y}$ which rotates the vector around the y -axis. Hence

$$
\begin{equation*}
M_{y} M_{z} \hat{J} \equiv \hat{J}=\left(0,0, \hat{J}_{z}\right), \tag{2.24}
\end{equation*}
$$

where

$$
\begin{align*}
& M_{z}=\left(\begin{array}{ccc}
\cos \gamma_{z} & -\sin \gamma_{z} & 0 \\
\sin \gamma_{z} & \cos \gamma_{z} & 0 \\
0 & 0 & 1
\end{array}\right)  \tag{2.25}\\
& M_{y}=\left(\begin{array}{ccc}
\cos \gamma_{y} & 0 & \sin \gamma_{y} \\
0 & 1 & 0 \\
-\sin \gamma_{y} & 0 & \cos \gamma_{y}
\end{array}\right) \tag{2.26}
\end{align*}
$$

Letting $Z_{\alpha} \equiv M_{y} M_{z} w_{\alpha}, w_{\alpha} \equiv\left(M_{y} M_{z}\right)^{T} z_{\alpha}$ gives the aligned system

$$
\begin{gather*}
\dot{Z}_{\alpha}=\frac{1}{4 \Pi} \sum_{\beta=1}^{N}{ }^{\prime} \Gamma_{\beta} \frac{Z_{\beta} \times Z_{\alpha}}{\left(1-Z_{\alpha} \cdot Z_{\beta}\right)}, Z_{\alpha}(0)=M_{y} M_{z} X_{\alpha}(0)  \tag{2.27}\\
\hat{J}=\sum_{\alpha=1}^{N} \Gamma_{\alpha} Z_{\alpha}=\left(0,0, \hat{J}_{z}\right) . \tag{2.28}
\end{gather*}
$$

Where

$$
\tilde{J}_{z}=\hat{J}_{z}=J_{z}=\text { const } .
$$



Figure 6. Rigid configurations on the rotating sphere are made up of rotations around two independent axes, with two independent frequencies. The vortices are layered on constant latitudinal planes that are perpendicular to the J axis.

### 2.3 Integrability

The relation between solutions of the original rotating system, $\vec{X}_{\alpha}(t)$, and solutions of the aligned system $Z_{\alpha}(t)$, and in some sense the key result of this chapter, is via the linear operator $\mathfrak{J}_{\Omega}^{J}(t)=M_{\Omega}(t) M_{z}^{-1} M_{y}^{-1}$

$$
\begin{equation*}
\vec{X}_{\alpha}(t)=\mathfrak{J}_{\Omega}^{J}(t) \vec{Z}_{\alpha}(t) \tag{2.29}
\end{equation*}
$$

This operator is time-dependent, but more importantly contains information on the original alignment of the $\mathbf{J}$ vector with the axis of rotation. Central to the question of integrability is the rate of separation of the vortices as measured by $\left\|\vec{X}_{\alpha}-\vec{X}_{\beta}\right\|^{2}$ on the rotating sphere and $\left\|\vec{Z}_{\alpha}-\vec{Z}_{\beta}\right\|^{2}$ on the non-rotating aligned sphere. The two quantities are equal since

$$
\begin{aligned}
\left\|\vec{X}_{\alpha}-\vec{X}_{\beta}\right\|^{2} & =\left\langle\vec{X}_{\alpha}-\vec{X}_{\beta}, \vec{X}_{\alpha}-\vec{X}_{\beta}\right\rangle=2\left(1-\left\langle\vec{X}_{\alpha}, \vec{X}_{\beta}\right\rangle\right) \\
& =2\left(1-\left\langle M_{\Omega}(t) M_{z}^{-1} M_{y}^{-1} \vec{Z}_{\alpha}, M_{\Omega}(t) M_{z}^{-1} M_{y}^{-1} \vec{Z}_{\beta}\right\rangle\right) \\
& =2\left(1-\left\langle M_{y} M_{z} M_{\Omega}(t)^{-1} M_{\Omega}(t) M_{z}^{-1} M_{y}^{-1} \vec{Z}_{\alpha}, \vec{Z}_{\beta}\right\rangle\right) \\
& =2\left(1-\left\langle\vec{Z}_{\alpha}, \vec{Z}_{\beta}\right\rangle\right)=\left\|\vec{Z}_{\alpha}-\vec{Z}_{\beta}\right\|^{2}
\end{aligned}
$$

For the aligned non-rotating system, we know from Kidambi \& Newton (1998) that the three-vortex problem is integrable for all vortex strengths. From this result and (2.29) follows:

Proposition 2.1. (Integrability on the rotating sphere) The three-vortex problem on the rotating sphere (2.1) is integrable for all vortex strengths. The four-vortex problem is integrable if the center of vorticity vector $\mathbf{J}=0$. All solutions on the rotating sphere are mapped to solutions on the aligned non-rotating sphere via the linear transformation (2.29).


Figure.7. one-frequency rigid Platonic solids on the rotating sphere oriented at angle $\gamma$ with respect to the north pole. (a) Tetrahedron (4 equal vortices): One vortex is placed at the top, three are evenly spaced around the ring at the base; (b) Octahedron ( 6 equal vortices): Four vortices are evenly spaced around the middle ring, one is placed at the top and one is placed at the bottom; (c) Hexahedron (8 equal vortices): Four vortices are evenly spaced around the top ring, four are evenly spaced around the bottom ring which is aligned with the top ring ; (d) Icosahedron (12 equal vortices): Five vortices are evenly spaced around the top ring, five vortices are evenly spaced on the bottom ring which is staggered with respect to the top ring, one vortex is placed at the top and one is placed at the bottom; (e) Dodecahedron (20 vortices): Five equal strength vortices ( $\Gamma_{1}=\Gamma \sin \theta_{1}$ ) are evenly spaced along the outer top ring, five equal strength vortices $\left(\Gamma_{1}=\Gamma \sin \theta_{1}\right)$ are evenly spaced along the outer bottom ring staggered with respect to the outer top ring, five equal strength vortices ( $\Gamma_{2}=\Gamma \sin \theta_{2}$ ) are evenly spaced along the inner top ring, five equal strength vortices $\left(\Gamma_{2}=\Gamma \sin \theta_{2}\right)$ are evenly spaced along the inner bottom ring staggered with respect to the inner top ring.

The proof for the non-rotating sphere can be found in Kidambi and Newton (1998), Borisov \& Pavlov (1998), Borisov \& Lebedev (1998), with discussions in Newton (2001). At first glance, this result is somewhat surprising in view of Noether's theorem and the fact that the rotating problem has one less conserved quantity than the non-rotating problem (e.g.
$J_{x}^{2}+J_{y}^{2}, J_{z}$ compared with $\left.J_{x}, J_{y}, J_{z}\right)$. However, the proof of integrability for the nonrotating problem relies solely on the conservation of the three independent and involutive quantities $J_{x}^{2}+J_{y}^{2}, J_{z}$, as well as the underlying Hamiltonian and never makes use of the fact that $J_{x}$ and $J_{y}$ are each conserved.

### 2.4 Rigid configurations

We now examine the evolution of rigid configurations on the rotating sphere, which we define as those in which distances between each pair of vortices remain fixed,i.e.
$\left\|\vec{X}_{\alpha}-\vec{X}_{\beta}\right\|^{2}=$ const. Note that since

$$
\begin{equation*}
\left\langle\vec{X}_{\alpha}, \vec{X}_{\beta}\right\rangle=\left\langle\vec{w}_{\alpha}, \vec{w}_{b}\right\rangle=\left\langle\vec{Z}_{\alpha}, \vec{Z}_{\beta}\right\rangle, \tag{2.30}
\end{equation*}
$$

configurations that are rigid on the non-rotating sphere (aligned or non-aligned) are also rigid on the rotating sphere, hence, in what follows, we will use equations (2.21) to draw conclusions regarding rigid configurations on the rotating sphere. In particular, taking the dot product of (2.23) with $\vec{w}_{\alpha}$ along with the condition that $\left\langle\vec{w}_{\alpha}, \vec{w}_{\beta}\right\rangle=$ const. gives

$$
\begin{equation*}
\vec{w}_{\alpha} \cdot \hat{J}=\text { const } . \tag{2.31}
\end{equation*}
$$

i.e. the angle between each vortex and the center of vorticity vector remains fixed. Next, using system (2.21) along with the ansatz that each vortex moves with the same frequency around the same axis, i.e. $\dot{\vec{w}}_{\alpha} \equiv \vec{\omega} \times \vec{w}_{\alpha}$, we obtain

$$
\begin{equation*}
\vec{\omega} \times \vec{w}_{\alpha}=\frac{1}{4 \Pi} \sum_{\beta=1}^{N}{ }^{\prime} \Gamma_{\beta} \frac{\vec{w}_{\beta} \times \vec{w}_{\alpha}}{\left(1-\vec{w}_{\alpha} \cdot \vec{w}_{\beta}\right)} \tag{2.32}
\end{equation*}
$$

Then multiply by $\Gamma_{\alpha}$ and summing over $\alpha$ gives the condition

$$
\begin{equation*}
\vec{\omega} \times \hat{J}=0 \tag{2.33}
\end{equation*}
$$

Thus, on the non-rotating sphere, non-degenerate $(\hat{J} \neq 0)$ rigid configurations that rotate around the J axis with frequency $\omega$ move on constant latitudinal planes perpendicular to J . Hence, on the rotating sphere we have:

Proposition 2.2. (Rigid configurations) Rigid configurations on the rotating sphere that rotate around the $\mathbf{J}$ axis with frequency $\omega$ move on a constant latitudinal planes perpendicular to $\mathbf{J}$. The center of vorticity vector $\mathbf{J}$ rotates around the $z$-axis with frequency $\Omega$ When $\omega=0$, the rigid configurations have one frequency $\Omega$, but in general they are made up of two independent frequencies $(\Omega, \omega)$. The general case is shown in figure 8 .

The one-frequency solutions are fixed equilibria on the non-rotating sphere, while the twofrequency solutions are relative equilibria on the aligned non-rotating sphere in view of the relation (2.29). Note also that it is sufficient to consider only the orientation range $0=\gamma=\pi$, as trajectories in the range $\pi<\gamma<2 \pi$ can be obtained by symmetry. In the region $0=\gamma<$ $\pi / 2$, the rigid body moves in the same direction as the solid-body rotation (eastward), whereas in the region $\pi / 2<\gamma=\pi$ it moves in the opposite direction (westward). In what follows, we will look at the representative values $\gamma=\pi / 4, \pi / 2,3 \pi / 4$.

### 2.4.1 One-frequency solution

To obtain necessary conditions for the one-frequency solutions on the rotating sphere, let $\dot{\vec{Z}}_{\alpha}=\vec{\omega} \times \vec{Z}_{\alpha}$ in (2.27), take the cross product with $\Gamma_{\alpha} \vec{Z}_{\alpha}$ and sum on $\alpha$,

$$
\begin{align*}
\sum_{\alpha=1}^{N} \Gamma_{\alpha} \vec{Z}_{\alpha} \times\left(\vec{\omega} \times \vec{Z}_{\alpha}\right) & =\frac{1}{4 \Pi} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N}{ }^{\prime} \Gamma_{\alpha} \Gamma_{\beta} \frac{\vec{Z}_{\alpha} \times\left(\vec{Z}_{\beta} \times \vec{Z}_{\alpha}\right)}{1-\vec{Z}_{\alpha} \cdot \vec{Z}_{\beta}} \\
& =\frac{1}{4 \Pi} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N}{ }^{\prime} \Gamma_{\alpha} \Gamma_{\beta} \frac{\vec{Z}_{\beta}-\vec{Z}_{\alpha}\left(\vec{Z}_{\alpha} \cdot \vec{Z}_{\beta}\right)}{1-\vec{Z}_{\alpha} \cdot \vec{Z}_{\beta}} \\
& =\frac{1}{4 \Pi} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \Gamma_{\alpha} \Gamma_{\beta} \vec{Z}_{\alpha} \\
& =\frac{1}{4 \Pi}\left(S \widetilde{J}-\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2} \vec{Z}_{\alpha}\right) \tag{2.34}
\end{align*}
$$



Figure.8. Dipole motion on a rotating sphere as a superposition of rotations about two axes with the two frequencies $\omega$ and $\Omega$ Orientation angle is given by $\gamma$.
where $S=\sum_{\alpha=1}^{N} \Gamma_{\alpha}$ is the total vorticity. Hence, a necessary condition for a fixed configuration ( $\omega=0$ ) on the aligned non-rotating sphere, i.e. a one-frequency rigid configuration on the rotating sphere is

$$
\begin{equation*}
S \widetilde{J}=\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2} \vec{Z}_{\alpha} \tag{2.35}
\end{equation*}
$$

It is interesting to note that the analogous condition for the existence of a fixed equilibrium configuration in the plane is given by

$$
\begin{equation*}
S^{2}=\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2} \tag{2.36}
\end{equation*}
$$

as described in Aref et al. (2003). We show in figure 7 a family of one-frequency solutions on the rotating sphere given by the Platonic solids oriented at angle $\gamma$ with respect to the axis of rotation. The details are described in the figure captions. Existence of these solutions as equilibria on the non-rotating sphere are described in Aref et al. (2003) and are special cases of some of the configurations studied in Lim et al. (2001) and Laurent-Polz (2002).We note that a recent result of Kurakin (2004) shows that on the non-rotating sphere the tetrahedron, octahedron, and icosahedron are nonlinearly stable, while the cube and dodecahedron are unstable. It is not clear whether the stability characteristics of these configurations are influenced by the application of $\mathfrak{J}_{\Omega}^{J}(t)$.

### 2.4.2 Two frequency solutions

To obtain necessary conditions for two-frequency rigid rotations on the rotating sphere, take the scalar product of equation (2.34) with $\omega$ :

$$
\begin{equation*}
\sum_{\alpha=1}^{N} \Gamma_{\alpha} \vec{\omega} \cdot\left(\vec{Z}_{\alpha} \times\left(\vec{\omega} \times \vec{Z}_{\alpha}\right)\right)=\frac{1}{4 \Pi}\left(\tilde{J} \cdot \vec{\omega}-\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2} \vec{Z}_{\alpha} \cdot \vec{\omega}\right) \tag{2.37}
\end{equation*}
$$



Figure.9. Wobbling mode: (a) Local ( $\beta$-plane) wobbling mode (after Hobson (1991)) associated with a vortex dipole; (b) Global wobbling mode associated with a vortex dipole moving eastward on the sphere with orientation $\gamma=\pi / 4$.


Figure.10. Tumbling mode: (a)Local ( $\beta$-plane) tumbling mode (after Hobson (1991)) associated with a vortex dipole; (b) Global tumbling mode associated with a vortex dipole moving westward on the sphere with orientation $\gamma=3 \pi / 4$.

Then use the fact that

$$
\begin{equation*}
\vec{\omega} \cdot\left(\vec{Z}_{\alpha} \times\left(\vec{\omega} \times \vec{Z}_{\alpha}\right)\right)=\left(\vec{Z}_{\alpha} \cdot \vec{Z}_{\alpha}\right)(\vec{\omega} \cdot \vec{\omega})-\left(\vec{\omega} \cdot \vec{Z}_{\alpha}\right)^{2} \tag{2.38}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\sum_{\alpha=1}^{N} \Gamma_{\alpha}\left(\left\|\vec{Z}_{\alpha}\right\|^{2}\|\vec{\omega}\|^{2}-\|\vec{\omega}\|^{2} \cos ^{2} \theta_{\alpha}\right)=\frac{1}{4 \Pi}\left(\tilde{J} \cdot \vec{\omega}-\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2}\|\vec{\omega}\| \cos \theta_{\alpha}\right) \tag{2.39}
\end{equation*}
$$

Where $\vec{Z}_{\alpha} \cdot \omega=\vec{\omega} \cos \theta_{\alpha}$. This gives a formula for the rotational frequency

$$
\begin{equation*}
\|\vec{\omega}\|^{2} \sum_{\alpha=1}^{N} \Gamma_{\alpha} \sin ^{2} \theta_{\alpha}=\frac{1}{4 \Pi}\left(S \widetilde{J} \cdot \vec{\omega}-\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2} \cos \theta_{\alpha}\right) \tag{2.40}
\end{equation*}
$$



Figure 11. Dipole oriented at angle $\pi / 4$, Northern and Southern hemispheres. Trajectory of $J$ vector is shown in the Northern hemisphere (small dashed circle). (a) 1:1 frequency ratio; (b) $2: 1$ frequency ratio; (c) $3: 1$ frequency ratio.

The analogous formula for rotating relative equilibria in the plane is

$$
\begin{equation*}
\|\vec{\omega}\| \sum_{\alpha=1}^{N} \Gamma_{\alpha}\left\|\vec{Z}_{\alpha}\right\|^{2}=\frac{1}{4 \Pi}\left(S^{2}-\sum_{\alpha=1}^{N} \Gamma_{\alpha}^{2}\right) \tag{2.41}
\end{equation*}
$$

described in Aref et al. (2003). Several examples of two-frequency solutions on the rotating sphere are described next.

### 2.4.3 Dipole dynamics

A vortex dipole, shown in figure 8 , is made up of two equal and opposite strength vortices, $\Gamma_{1}=\Gamma, \Gamma_{2}=-\Gamma, \theta_{1}=\theta, \theta_{2}=\Pi-\theta, S=0$. The convention in this figure and those that follow is that black point vortices are positive (i.e. counterclockwise circulation) while white ones are negative (i.e. clockwise circulation). Formula (2.40) then becomes

$$
\begin{equation*}
\omega=\frac{\Gamma}{8 \Pi \cos \theta} \tag{2.42}
\end{equation*}
$$

It is a fundamental result that on a non-rotating sphere, a dipole follows the geodesic (i.e. great circle) that perpendicularly bisects the geodesic segment that connects the two vortices (Kimura (1999)). Motion of a dipole on the sphere for the two-way coupled model was carried out in DiBattista \& Polvani (1998) as an initial value problem in which the background vorticity (i.e. all vorticity not associated with the dipole)


Figure.12. Dipole oriented at angle $\pi / 2$, Northern and Southern hemispheres. (a) 1:1 frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.
is placed initially in constant latitudinal strips in order to model solid body rotation. Both point vortex dipoles and distributed vortex dipoles were tracked numerically, showing, among other things, that dipoles no longer follow geodesic paths, but move on more complicated trajectories and in some instances can lose stability and tear apart. Because they only consider the case in which the $\mathbf{J}$ vector is aligned with the axis of rotation, they cannot distinguish between the effects of mis-alignment and the effect of coupling to the background field. Dipole motion on the $\beta$-plane in a one-way coupled model was studied by Hobson (1991) where two modes of motion, 'tumbling' and 'wobbling', were identified. Similarly, a one-way coupled $\beta$-plane model and the related modon solution were studied in Matsuoka \& Nozaki (1992). Our general configuration is shown in figure 8 and is governed by three key parameters. The orientation angle, $\gamma$, measures the angle between the $J$ vector and the axis of rotation around the North Pole. The frequency $\Omega$ is associated with the solidbody rotation, while frequency $\omega$ is associated with the dipole motion in the absence of rotation, i.e. its frequency around a great circle as given by formula (2.42). This is determined by the choice of vortex strengths, which we take as $\Gamma_{1}=1$ and $\Gamma_{2}=-1$ and the dipole separation (chord distance), which we take as $d=0.1=\sin \theta$. In all cases, we take the initial center point of the dipole to lie on the equator at the front of the sphere (defined as longitude $\varphi=0$ ), as shown in the figure. Figure 9 shows the $\beta$-plane wobbling mode of Hobson (1991) (figure 9(a)) and the corresponding 'global' wobbling mode (figure 9(b)) on the full sphere. The tumbling modes are shown in figures $10(\mathrm{a})$,(b). What distinguishes the two cases is the orientation angle. When $0<\gamma<\pi / 2$, the dipole moves in the same direction as the rotation (eastward) and produces a wobbling trajectory. When $\pi / 2<\gamma<\pi$, it moves initially opposite the direction of rotation (westward) and produces a tumbling trajectory.

Figures 11,12 , and 13 show the dipole trajectories on the rotating sphere with orientation angles $\pi / 4, \pi / 2$, and $3 \pi / 4$ respectively. When the frequency ratio $\omega / \Omega$ is rational, the motion is periodic. Cases with frequency ratios $\omega / \Omega=1,2,3$ are shown in the Northern and Southern hemispheres, along with the trajectory of the $J$ vector (dashed circle). Note that the cases $\gamma=\pi / 4$ (figure 11) and $\gamma=3 \pi / 4$ (figure 13) are not related to each other via symmetries. In the first case, the dipole moves initially in the same direction as the solidbody rotation, while in the second case, it moves opposite to the direction of the solid-body rotation. When the frequency ratio is irrational, the long-time trajectory densely covers the available surface of the sphere allowed by the choice of the angle of orientation $\gamma$.


Figure.13. Dipole oriented at angle $3 \pi / 4$, Northern and Southern hemispheres. Trajectory of $J$ vector is shown in the Southern hemisphere (small dashed circle). (a) $1: 1$ frequency ratio; (b) $2: 1$ frequency ratio; (c) $3: 1$ frequency ratio.

### 2.4.4 Rings

In the case of an isolated ring in which N -vortices of equal strength are arranged around a constant latitude cap perpendicular to $\mathbf{J}$ as shown in figure 14, we have $\Gamma_{\alpha}=\Gamma, \theta_{\alpha}=\theta$, $S=N \Gamma$


Figure.14. Equal strength vortices evenly spaced on a constant latitude cap perpendicular to the center of vorticity vector on the rotating sphere.
and $\widetilde{J}_{z}=N \Gamma \cos \theta$ and formula (2.40) reduces to

$$
\begin{equation*}
\omega=\frac{\Gamma(N-1)}{4 \Pi} \frac{\cos \theta}{\sin ^{2} \theta} \tag{2.43}
\end{equation*}
$$

The stability of such configurations on the non-rotating sphere (as well as ones with an additional polar vortex) have been studied in Dritschel \& Polvani (1993), Cabral et al. (2003), and Laurent-Polz et al. (2004), and it is known that a single ring made up of $N$ equal strength, evenly spaced point vortices is unstable for all co-latitudes if $N=7$, whereas for $N$ $<7$ there exist ranges of Lyapunov stability when the ring is near a pole. In general terms, an additional polar vortex can serve to stabilize or destabilize a ring, hence it stands to reason that the addition of solid-body rotation $(\Omega \neq 0)$ may also alter the stability property of the ring, although this question has not been addressed. Figure 15 shows the vortex paths of a four-vortex ring with orientation angle $\pi / 4$. The ring radius is $r=0.1$ and the trajectory of the $J$ vector is shown as the dashed circle. Frequency ratios of $1: 1,2: 1$, and $3: 1$ are shown in the Northern hemispheres as none of the vortices crosses the equator. Figure 13 shows the same ring oriented at angle $\gamma=\pi / 2$. Trajectories corresponding to frequency ratios of $1: 1,2$ $: 1$, and $3: 1$ are shown from the perspective of the front of the sphere. Figure.17. shows the same ring oriented at angle $\gamma=3 \pi / 4$. In this case, the ring's motion is opposite to the direction of rotation and gives different trajectories than those shown in figure.15. Frequency ratios of $1: 1,2: 1$, and $3: 1$ are shown in the Southern hemispheres as none of the vortices crosses the equator.

### 2.4.5 Stacked rings: The platonic solids

More complex two-frequency rigid configurations on the rotating sphere are given by the Platonic solids shown in figure 18, where the vorticities have both positive and negative signs. Details are given in the figure captions. We show the evolution of a two-frequency tetrahedron in figures 19, 20, 21. In particular, figure 19 shows the trajectories of the four vortices making up a tetrahedral configuration oriented at angle $\gamma=\pi / 4$. The dashed curve marks the trajectory of the top vortex, which in this case stays in the Northern hemisphere. Frequency ratios of $1: 1,2: 1$, and $3: 1$ are shown. Figure 20 shows the same configuration
oriented at angle $\gamma=\pi / 2$ with frequency ratios of $1: 1,2: 1$, and $3: 1$. The top of the tetrahedron in this case stays along the equator. Finally, figure 18 shows the tetrahedral configuration oriented at angle $\gamma=3 \pi / 4$ with frequency ratios $1: 1,2: 1$, and $3: 1$. The top of the configuration now moves along the dashed circular curve shown in the Southern hemisphere.


Figure.15. Four vortex ring oriented at angle $\pi / 4$. Dashed circle shows the trajectory of the $J$ vector. (a) $1: 1$ frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.


Figure.16. Four vortex ring oriented at angle $\pi / 2$. Front of sphere is shown, dashed curve is the path of the $J$ vector. (a) 1:1 frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.


Figure.17. Four vortex ring oriented at angle $3 \pi / 4$, Southern hemisphere only. Dashed curve is the path of the $J$ vector. (a) 1:1 frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.


Figure.18. Some examples of vortices on relative equilibria configuration: (a) Tetrahedron: $\Gamma$ is placed at the top, $-\Gamma$ are evenly spaced around the bottom ring; (b) Octahedron: $\Gamma$ 's are evenly spaced around the middle ring, another is placed at the top, $-\Gamma$ is placed at the bottom; (c) Hexahedron: $\Gamma$ 's are evenly spaced around the top ring, $-\Gamma$ 's are evenly spaced around the bottom ring; (d) Icosahedron: $\Gamma$ 's are evenly spaced on the top ring $-\Gamma$ 's are evenly spaced on the bottom ring, another $\Gamma$ is placed at the top and $-\Gamma$ is placed at the bottom; (e) Dodecahedron: $\Gamma_{1}=\Gamma \sin \theta_{1}$ are evenly spaced along the outer top ring, $\Gamma_{1}=-\Gamma \sin \theta_{1}$ are evenly spaced along the outer bottom ring staggered with respect to the outer top ring, $\Gamma_{2}=\Gamma \sin \theta_{2}$ are evenly spaced along the inner top ring, $\Gamma_{2}=-\Gamma \sin \theta_{2}$ are evenly spaced along the inner bottom ring staggered with respect to the inner top ring.


Figure.19. Two-frequency tetrahedron oriented at angle $\pi / 4$, Northern and Southern hemispheres. Dashed circle in the Northern hemisphere marks the top of the configuration. (a) 1:1 frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.


Figure.20. Two-frequency tetrahedron oriented at angle $\pi / 2$, Northern and Southern hemispheres. (a) $1: 1$ frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.


Figure.21. Two-frequency tetrahedron oriented at angle $3 \pi / 4$, Northern and Southern hemispheres. Dashed circle in the Southern hemisphere marks the top of the configuration. (a) 1:1 frequency ratio; (b) 2:1 frequency ratio; (c) 3:1 frequency ratio.

## Chapter 3

## Numerically solving the $\mathbf{N}$-vortex problem on a sphere

## 3-1 Numerical solution in Cartesian coordinates:

In chapter 1, formulas for calculating the effects of one vortex on another and velocity vector for each vortex have been studied by thoroughly. Now it is time to expand these solutions to the N -vortex problem and find an accurate numerical solution specifically for the case of N interacting dipoles. As discussed in chapter 1 , the effect of vortex $\alpha$ with strength $\Gamma_{\alpha}$ on velocity field of vortex $\beta$ when both of them are located on a sphere by radius R is as follow:

$$
\stackrel{\vec{X}}{\beta}=\frac{\Gamma_{\alpha}}{4 \Pi R} \cdot \frac{\vec{X}_{\alpha} \times \vec{X}_{\beta}}{R^{2}-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}}
$$

Now is time to expand this solution to the case that we want to calculate effects of n-vortex like $\alpha$ with strength $\Gamma_{\alpha}$ on vortex $\beta$ :

$$
\begin{equation*}
\stackrel{\rightharpoonup}{X}_{\beta}=\frac{1}{4 \Pi R} \sum_{\beta \neq \alpha}^{N} \Gamma_{\alpha} \cdot \frac{\vec{X}_{\alpha} \times \vec{X}_{\beta}}{R^{2}-\vec{X}_{\alpha} \cdot \vec{X}_{\beta}} \tag{3.1}
\end{equation*}
$$

$\vec{X}_{\alpha}$ is a vector which shows position of vortex $\alpha$ at any point, as we are writing formulas in a Cartesian coordinate system each vector has three elements relate to basis of Cartesian coordinates $(\mathrm{i}, \mathrm{j}, \mathrm{k})$. To calculate the solutions to (3.1) numerically, we revise the previous formulas as follows:

$$
\begin{equation*}
\vec{X}_{\alpha}=X_{\alpha} \hat{i}+Y_{\alpha} \hat{j}+Z_{\alpha} \hat{k} \Rightarrow \dot{\vec{X}}=\dot{X}_{\alpha} \hat{i}+\dot{Y}_{\alpha} \hat{j}+\dot{Z}_{\alpha} \hat{k} \tag{3.2}
\end{equation*}
$$

$$
\begin{gather*}
\dot{X}_{\beta} \hat{i}+\dot{Y}_{\beta} \hat{j}+\dot{Z}_{\beta} \hat{k}=\frac{\Gamma_{\alpha}}{4 \Pi R} \cdot \frac{\left(Y_{\alpha} Z_{\beta}-Z_{\alpha} Y_{\beta}\right) \hat{i}-\left(X_{\alpha} Z_{\beta}-Z_{\alpha} X_{\beta}\right) \hat{j}+\left(X_{\alpha} Y_{\beta}-Y_{\alpha} X_{\beta}\right) \hat{k}}{\left(R^{2}-X_{\alpha} X_{\beta}-Y_{\alpha} Y_{\beta}-Z_{\alpha} Z_{\beta}\right)}  \tag{3.3}\\
\dot{X}_{\beta}=\frac{\Gamma_{\alpha}}{4 \Pi R} \cdot \frac{\left(Y_{\alpha} Z_{\beta}-Z_{\alpha} Y_{\beta}\right)}{\left(R^{2}-X_{\alpha} X_{\beta}-Y_{\alpha} Y_{\beta}-Z_{\alpha} Z_{\beta}\right)}  \tag{3.4}\\
\dot{Y}_{\beta}=\frac{\Gamma_{\alpha}}{4 \Pi R} \cdot \frac{-\left(X_{\alpha} Z_{\beta}-Z_{\alpha} X_{\beta}\right) \hat{j}}{\left(R^{2}-X_{\alpha} X_{\beta}-Y_{\alpha} Y_{\beta}-Z_{\alpha} Z_{\beta}\right)}  \tag{3.5}\\
\dot{Z}_{\beta}=\frac{\Gamma_{\alpha}}{4 \Pi R} \cdot \frac{\left(X_{\alpha} Y_{\beta}-Y_{\alpha} X_{\beta}\right) \hat{k}}{\left(R^{2}-X_{\alpha} X_{\beta}-Y_{\alpha} Y_{\beta}-Z_{\alpha} Z_{\beta}\right)} \tag{3.6}
\end{gather*}
$$

Now if assume $\mathrm{R}=1$ and we have N distinct vortices located on the unit sphere then:

$$
\begin{align*}
& \dot{X}_{i}=\frac{1}{4 \Pi} \cdot \sum_{j \neq i}^{N} \Gamma_{j} \cdot \frac{\left(Y_{j} Z_{i}-Z_{j} Y_{i}\right)}{\gamma_{i j}}  \tag{3.7}\\
& \dot{Y}_{i}=\frac{-1}{4 \Pi} \cdot \sum_{j \neq i}^{N} \Gamma_{j} \cdot \frac{\left(X_{j} Z_{i}-Z_{j} X_{i}\right)}{\gamma_{i j}}  \tag{3.8}\\
& \dot{Z}_{i}=\frac{1}{4 \Pi} \cdot \sum_{j \neq i}^{N} \Gamma_{j} \cdot \frac{\left(X_{j} Y_{i}-Y_{j} X_{i}\right)}{\gamma_{i j}} \tag{3.9}
\end{align*}
$$

Where $\gamma_{i j}=1-X_{i} X_{j}-Y_{i} Y_{j}-Z_{i} Z_{j}$

We now consider the special case where 2 N vortices are grouped into pairs of N dipoles. For N dipoles with different strengths there will be 2 N vortices and if assume vortices i and $\mathrm{i}+\mathrm{N}$ are pairs of a dipole then $\Gamma_{i}=-\Gamma_{i+N}$, and equations (3.7) thru (3.9) can be modified as follow:

$$
\begin{align*}
& \dot{X}_{i}=\frac{1}{4 \Pi} \cdot\left(\sum_{j \neq i}^{N} \Gamma_{j} \cdot \frac{\left(Y_{j} Z_{i}-Z_{j} Y_{i}\right)}{\gamma_{i j}}-\sum_{j+n \neq i}^{N} \Gamma_{j} \cdot \frac{Y_{j+N} Z_{i}-Z_{j+N} Y_{i}}{\gamma_{i(j+N)}}\right)  \tag{3.10}\\
& \dot{Y}_{i}=\frac{1}{4 \Pi} \cdot\left(\sum_{j \neq i}^{N}-\Gamma_{j} \cdot \frac{\left(X_{j} Z_{i}-Z_{j} X_{i}\right)}{\gamma_{i j}}+\sum_{j+n \neq i}^{N} \Gamma_{j} \cdot \frac{X_{j+N} Z_{i}-Z_{j+N} X_{i}}{\gamma_{i(j+N)}}\right)  \tag{3.11}\\
& \dot{Z}_{i}=\frac{1}{4 \Pi} \cdot\left(\sum_{j \neq i}^{N} \Gamma_{j} \cdot \frac{\left(X_{j} Y_{i}-Y_{j} X_{i}\right)}{\gamma_{i j}}-\sum_{j+n \neq i}^{N} \Gamma_{j} \cdot \frac{X_{j+N} Y_{i}-Y_{j+N} X_{i}}{\gamma_{i(j+N)}}\right) \tag{3.12}
\end{align*}
$$

### 3.2 Numerical solution in J-vector coordinates:

In Cartesian coordinates for N dipoles, there are 2 N equations which must be solved simultaneously. To follow each dipole's trajectory directly, we introduce two new vectors $\vec{J}_{i}$ and $\vec{J}_{i+N}$ instead of $\vec{X}_{i}$ and $\vec{X}_{i+N}$. If assume that $\vec{X}_{i}, \vec{X}_{i+N}$ form a dipole with strength $\Gamma_{i}$ then relation between J and X vectors can be defined as follow:

$$
\begin{gather*}
\vec{J}_{i}=\Gamma_{i} \vec{X}_{i}+\Gamma_{i+N} \vec{X}_{i+N} \\
\vec{J}_{i+n}=\Gamma_{i} \vec{X}_{i}-\Gamma_{i+N} \vec{X}_{i+N} \tag{3.13}
\end{gather*}
$$

$\vec{X}_{i}, \vec{X}_{i+N}$ form a dipole therefore $\Gamma_{i}=-\Gamma_{i+N}$ then:

$$
\begin{gather*}
\vec{J}_{i}=\Gamma_{i}\left(\vec{X}_{i}-\vec{X}_{i+N}\right)  \tag{3.14}\\
\vec{J}_{i+n}=\Gamma_{i}\left(\vec{X}_{i}+\vec{X}_{i+N}\right)
\end{gather*}
$$

The vectors $\vec{J}_{1}, \vec{J}_{2} \cdots \vec{J}_{N} ; \vec{J}_{N+1}, \cdots \vec{J}_{2 N}$ form a new coordinate system we call it Dipole Coordinate system. They are related to the Cartesian coordinates as follow:

$$
\begin{align*}
\vec{X}_{i} & =\frac{\left(\vec{J}_{i}-\vec{J}_{i+N}\right)}{2 \Gamma_{i}}  \tag{3.15}\\
\vec{X}_{i+N} & =\frac{\left(-\vec{J}_{i}+\vec{J}_{i+N}\right)}{2 \Gamma_{i}}
\end{align*}
$$

Or in matrix form: $\hat{J}=M \hat{X}$ where $\widehat{J} \equiv\left(\begin{array}{c}\vec{J}_{1} \\ \vdots \\ \vec{J}_{N} \\ \vec{J}_{N+1} \\ \vdots \\ \vec{J}_{2 N}\end{array}\right), \hat{X} \equiv\left(\begin{array}{c}\vec{X}_{1} \\ \vdots \\ \vec{X}_{N} \\ \vec{X}_{N+1} \\ \vdots \\ \vec{X}_{2 N}\end{array}\right)$ and M is a $2 N \times 2 N$
Matrix with block-diagonal structure:

$$
M \equiv\left(\begin{array}{cc}
M_{\Gamma} & -M_{\Gamma} \\
M_{\Gamma} & M_{\Gamma}
\end{array}\right) \text { where } M_{\Gamma} \in R^{N} \times R^{N} \text { and } M_{\Gamma} \equiv\left(\begin{array}{ccc}
\Gamma_{1} & & 0 \\
& \ddots & \\
0 & & \Gamma_{N}
\end{array}\right)
$$

In the special case where the N dipoles are located on a unit sphere $(\mathrm{R}=1), \vec{J}_{i}$ and $\vec{J}_{i+N}$ are perpendicular to one another.

$$
\vec{J}_{i} \cdot \vec{J}_{i+N}=\left(\vec{X}_{i}+\vec{X}_{i+N}\right) \cdot\left(\vec{X}_{i}-\vec{X}_{i+N}\right)=\left\|\vec{X}_{i}\right\|^{2}-\vec{X}_{i} \cdot \vec{X}_{i+N}+\vec{X}_{i+N} \cdot \vec{X}_{i}-\left\|\vec{X}_{i+N}\right\|^{2}=0
$$

Now $\vec{J}_{i}$ and $\vec{J}_{i+N}$ are two perpendicular vectors originating at the center of the sphere, one of them $\left(\vec{J}_{i}\right)$ clearly shows the trajectory of dipoles and the other one $\left(\vec{J}_{i+N}\right)$ represent direction of rotation and the vector around which dipoles are rotating.

### 3.3 N -vortex problem in dipole coordinate (special case: $\mathrm{N}=1$, one dipole on unit sphere)

$$
\begin{gathered}
\stackrel{\rightharpoonup}{X} \\
1
\end{gathered}=\frac{\Gamma_{2}}{4 \Pi} \cdot \frac{\vec{X}_{2} \times \vec{X}_{1}}{R^{2}-\vec{X}_{1} \cdot \vec{X}_{2}}, \vec{X}_{2}=\frac{\Gamma_{1}}{4 \Pi} \cdot \frac{\vec{X}_{1} \times \vec{X}_{2}}{R^{2}-\vec{X}_{1} \cdot \vec{X}_{2}}, \Gamma_{1}=-\Gamma_{2}, ~=\vec{J}_{1}\left(\vec{X}_{1}-\vec{X}_{2}\right), \vec{J}_{2}=\Gamma_{2}\left(\vec{X}_{1}+\vec{X}_{2}\right) \text { and } \hat{J}=M \hat{X} \text { therefore: }
$$

$$
\begin{gathered}
M \equiv\left(\begin{array}{cc}
\Gamma_{1} & -\Gamma_{1} \\
\Gamma_{1} & \Gamma_{1}
\end{array}\right) \\
\vec{X}_{1}=\frac{\left(\vec{J}_{1}+\vec{J}_{2}\right)}{2 \Gamma_{1}}, \vec{X}_{2}=\frac{-\left(\vec{J}_{1}-\vec{J}_{2}\right)}{2 \Gamma_{1}}
\end{gathered}
$$

Now we would like to transfer (3.1) to dipole coordinates for the special case $\mathrm{N}=1$ :

$$
\begin{gather*}
\vec{X}_{1} \times \vec{X}_{2}=-\frac{1}{4 \Gamma_{1}^{2}}\left(\vec{J}_{1}+\vec{J}_{2}\right) \times\left(\vec{J}_{1}-\vec{J}_{2}\right)=-\frac{1}{4 \Gamma_{1}^{2}}\left(\vec{J}_{1} \times \vec{J}_{1}-\vec{J}_{1} \times \vec{J}_{2}+\vec{J}_{2} \times \vec{J}_{1}-\vec{J}_{2} \times \vec{J}_{2}\right) \\
\quad \Rightarrow \vec{X}_{1} \times \vec{X}_{2}=\frac{1}{2 \Gamma_{1}^{2}} \vec{J}_{1} \times \vec{J}_{2}  \tag{3.16}\\
\vec{X}_{1} \cdot \vec{X}_{2}=-\frac{1}{4 \Gamma_{1}^{2}}\left(\vec{J}_{1}+\vec{J}_{2}\right) \cdot\left(\vec{J}_{1}-\vec{J}_{2}\right)=-\frac{1}{4 \Gamma_{1}^{2}}\left(\left\|\vec{J}_{1}\right\|^{2}-\vec{J}_{1} \cdot \vec{J}_{2}+\vec{J}_{2} \cdot \vec{J}_{1}-\left\|\vec{J}_{2}\right\|^{2}\right) \\
\Rightarrow \vec{X}_{1} \cdot \vec{X}_{2}=\frac{1}{4 \Gamma_{1}^{2}}\left(\left\|\vec{J}_{1}\right\|^{2}-\left\|\vec{J}_{2}\right\|^{2}\right) \tag{3.17}
\end{gather*}
$$

Now Equation (3.1) for the special case of $\mathrm{N}=1$ yields:

$$
\begin{align*}
& \dot{J}_{1}+\dot{J}_{2}=\frac{\Gamma_{1}^{2}}{\Pi} \cdot \frac{\vec{J}_{1} \times \vec{J}_{2}}{\left(4 \Gamma_{1}^{2}+\left\|\vec{J}_{1}\right\|^{2}-\left\|\vec{J}_{2}\right\|^{2}\right)} \\
& \dot{J}_{1}-\dot{J}_{2}=-\frac{\Gamma_{1}^{2}}{\Pi} \cdot \frac{\vec{J}_{1} \times \vec{J}_{2}}{\left(4 \Gamma_{1}^{2}+\left\|\vec{J}_{1}\right\|^{2}-\left\|\vec{J}_{2}\right\|^{2}\right)} \tag{3.18}
\end{align*}
$$

Which when added and subtracted together result the equations in dipole coordinates:

$$
\begin{align*}
& \dot{\vec{J}}_{1}=0 \\
& \dot{\vec{J}}_{2}=\frac{\Gamma_{1}^{2}}{\Pi} \cdot \frac{\vec{J}_{1} \times \vec{J}_{2}}{\left(4 \Gamma_{1}^{2}+\left\|J_{1}\right\|^{2}-\left\|J_{2}\right\|^{2}\right)}=\frac{1}{4 \Pi} \cdot \frac{\vec{J}_{1} \times \vec{J}_{2}}{\left(1+\frac{\left\|J_{1}\right\|^{2}-\left\|J_{2}\right\|^{2}}{4 \Gamma_{1}}\right)} \tag{3.19}
\end{align*}
$$

From (3.19) we conclude that $\vec{J}_{1}$ and $\left\|\vec{J}_{2}\right\|$ are constant and $\vec{J}_{2}$ rotates around $\vec{J}_{1}$ with frequency $\Omega \equiv\left\|\vec{J}_{1}\right\|$ so we have two uncoupled equation that each of them shows trajectory of a J vector independently.

### 3.4 N -vortex problem in dipole coordinate (special case: $\mathrm{N}=2$, two dipole on unit sphere)

For simplicity if we define a new variable R as follow:

$$
\begin{equation*}
\vec{R}_{m, n}=\frac{\vec{X}_{m} \times \vec{X}_{n}}{\left(1-\vec{X}_{m} \cdot \vec{X}_{n}\right)} \tag{3.20}
\end{equation*}
$$

Then we will have

$$
\begin{aligned}
& \dot{\vec{X}}_{1}=\frac{\Gamma_{2}}{4 \Pi} \cdot \vec{R}_{2,1}+\frac{\Gamma_{3}}{4 \Pi} \cdot \vec{R}_{3,1}+\frac{\Gamma_{4}}{4 \Pi} \cdot \vec{R}_{4,1} \\
& \dot{\vec{X}}_{2}=\frac{\Gamma_{1}}{4 \Pi} \cdot \vec{R}_{1,2}+\frac{\Gamma_{3}}{4 \Pi} \cdot \vec{R}_{3,2}+\frac{\Gamma_{4}}{4 \Pi} \cdot \vec{R}_{4,2} \\
& \dot{\vec{X}}_{3}=\frac{\Gamma_{1}}{4 \Pi} \cdot \vec{R}_{1,3}+\frac{\Gamma_{2}}{4 \Pi} \cdot \vec{R}_{2,3}+\frac{\Gamma_{4}}{4 \Pi} \cdot \vec{R}_{4,3} \\
& \dot{\vec{X}}_{4}=\frac{\Gamma_{1}}{4 \Pi} \cdot \vec{R}_{1,4}+\frac{\Gamma_{2}}{4 \Pi} \cdot \vec{R}_{2,4}+\frac{\Gamma_{3}}{4 \Pi} \cdot \vec{R}_{3,1}
\end{aligned}
$$

As we now $\widehat{J}=\left[\begin{array}{cccc}\Gamma_{1} & 0 & -\Gamma_{1} & 0 \\ 0 & \Gamma_{2} & 0 & -\Gamma_{2} \\ \Gamma_{1} & 0 & \Gamma_{1} & 0 \\ 0 & \Gamma_{2} & 0 & \Gamma_{2}\end{array}\right] \hat{X}$ and we can transfer $\vec{X}$ and $\vec{R}$ to dipole
coordinate system as follow:

$$
\begin{align*}
& \vec{X}_{1}=\frac{\vec{J}_{1}+\vec{J}_{3}}{2 \Gamma_{1}}, \vec{X}_{2}=\frac{\vec{J}_{2}+\vec{J}_{4}}{2 \Gamma_{2}}, \vec{X}_{3}=\frac{-\vec{J}_{1}+\vec{J}_{3}}{2 \Gamma_{1}}, \vec{X}_{4}=\frac{-\vec{J}_{2}+\vec{J}_{4}}{2 \Gamma_{2}} \\
& \dot{\vec{J}}_{1}=\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(-\vec{R}_{1,2}-\vec{R}_{3,4}-\vec{R}_{2,3}+\vec{R}_{1,4}\right) \\
& \dot{\vec{J}}_{2}=\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(\vec{R}_{1,2}+\vec{R}_{3,4}+\vec{R}_{2,3}-\vec{R}_{1,4}\right) \\
& \dot{\vec{J}}_{3}=\frac{1}{4 \Pi} \cdot \frac{\vec{J}_{1} \times \vec{J}_{3}}{\left(1+\frac{\left\|\vec{J}_{1}\right\|^{2}-\left\|\vec{J}_{3}\right\|^{2}}{4 \Gamma_{1}^{2}}\right)}+\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(-\vec{R}_{1,2}+\vec{R}_{3,4}+\vec{R}_{2,3}+\vec{R}_{1,4}\right) \tag{3.21}
\end{align*}
$$

$$
\dot{\vec{J}}_{4}=\frac{1}{4 \Pi} \cdot \frac{\vec{J}_{2} \times \vec{J}_{4}}{\left(1+\frac{\left\|\vec{J}_{2}\right\|^{2}-\left\|\vec{J}_{4}\right\|^{2}}{4 \Gamma_{2}^{2}}\right)}+\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(\vec{R}_{1,2}-\vec{R}_{3,4}+\vec{R}_{2,3}+\vec{R}_{1,4}\right)
$$

Where:

$$
\begin{aligned}
& \vec{R}_{1,2}=\frac{\left(\vec{J}_{1} \times \vec{J}_{2}+\vec{J}_{1} \times \vec{J}_{4}-\vec{J}_{2} \times \vec{J}_{3}+\vec{J}_{3} \times \vec{J}_{4}\right)}{4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}+\vec{J}_{1} \cdot \vec{J}_{4}+\vec{J}_{2} \cdot \vec{J}_{3}+\vec{J}_{3} \cdot \vec{J}_{4}\right)} \\
& \vec{R}_{1,4}=\frac{\left(\vec{J}_{1} \times \vec{J}_{2}-\vec{J}_{1} \times \vec{J}_{4}-\vec{J}_{2} \times \vec{J}_{3}-\vec{J}_{3} \times \vec{J}_{4}\right)}{-4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}-\vec{J}_{1} \cdot \vec{J}_{4}+\vec{J}_{2} \cdot \vec{J}_{3}-\vec{J}_{3} \cdot \vec{J}_{4}\right)} \\
& \vec{R}_{2,3}=\frac{\left(-\vec{J}_{1} \times \vec{J}_{2}-\vec{J}_{1} \times \vec{J}_{4}-\vec{J}_{2} \times \vec{J}_{3}+\vec{J}_{3} \times \vec{J}_{4}\right)}{-4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}+\vec{J}_{1} \cdot \vec{J}_{4}-\vec{J}_{2} \cdot \vec{J}_{3}-\vec{J}_{3} \cdot \vec{J}_{4}\right)} \\
& \vec{R}_{1,2}=\frac{\left(\vec{J}_{1} \times \vec{J}_{2}-\vec{J}_{1} \times \vec{J}_{4}+\vec{J}_{2} \times \vec{J}_{3}+\vec{J}_{3} \times \vec{J}_{4}\right)}{4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}-\vec{J}_{1} \cdot \vec{J}_{4}-\vec{J}_{2} \cdot \vec{J}_{3}+\vec{J}_{3} \cdot \vec{J}_{4}\right)}
\end{aligned}
$$

## 3.5 $\mathbf{N}$-vortex problem in dipole coordinates (general case; $\mathbf{N}$ dipoles on unit sphere)

The general case of N interacting dipoles can be written as follows:

$$
\vec{R}_{m, n}=\frac{\vec{X}_{m} \times \vec{X}_{n}}{\left(1-\vec{X}_{m} \cdot \vec{X}_{n}\right)}, \vec{X}_{m}=\frac{\vec{J}_{m}+\vec{J}_{n}}{2 \Gamma_{m}}, \vec{X}_{m+N}=\frac{-\vec{J}_{m}+\vec{J}_{n}}{2 \Gamma_{m}}
$$

Therefore (3.1) when $\mathrm{R}=1$, can be re-written as:

$$
\begin{equation*}
\dot{\vec{X}}_{m}=\sum_{n \neq m}^{2 N} \frac{\Gamma_{n}}{4 \Pi} \cdot \vec{R}_{n, m} \quad, m \in[1, \cdots, 2 N] \tag{3.22}
\end{equation*}
$$

If $m, n<N$

$$
\vec{R}_{m, n}=\frac{\left(\vec{J}_{m} \times \vec{J}_{n}+\vec{J}_{m} \times \vec{J}_{n+N}+\vec{J}_{m+N} \times \vec{J}_{n}+\vec{J}_{m+N} \times \vec{J}_{n+N}\right)}{4 \Gamma_{m} \Gamma_{n}-\left(\vec{J}_{m} \cdot \vec{J}_{n}+\vec{J}_{m} \cdot \vec{J}_{n+N}+\vec{J}_{m+N} \cdot \vec{J}_{n}+\vec{J}_{m+N} \cdot \vec{J}_{n+N}\right)}
$$

If $m<N \wedge n>N \quad \vec{R}_{m, n}=\frac{\left(\vec{J}_{m} \times \vec{J}_{n}-\vec{J}_{m} \times \vec{J}_{n+N}+\vec{J}_{m+N} \times \vec{J}_{n}-\vec{J}_{m+N} \times \vec{J}_{n+N}\right)}{4 \Gamma_{m} \Gamma_{n}-\left(\vec{J}_{m} \cdot \vec{J}_{n}-\vec{J}_{m} \cdot \vec{J}_{n+N}+\vec{J}_{m+N} \cdot \vec{J}_{n}-\vec{J}_{m+N} \cdot \vec{J}_{n+N}\right)}$

If $m>N \wedge n<N \quad \vec{R}_{m, n}=\frac{\left(\vec{J}_{m} \times \vec{J}_{n}+\vec{J}_{m} \times \vec{J}_{n+N}-\vec{J}_{m+N} \times \vec{J}_{n}-\vec{J}_{m+N} \times \vec{J}_{n+N}\right)}{4 \Gamma_{m} \Gamma_{n}-\left(\vec{J}_{m} \cdot \vec{J}_{n}+\vec{J}_{m} \cdot \vec{J}_{n+N}-\vec{J}_{m+N} \cdot \vec{J}_{n}-\vec{J}_{m+N} \cdot \vec{J}_{n+N}\right)}$

If $m>N \wedge n>N \quad \vec{R}_{m, n}=\frac{\left(\vec{J}_{m} \times \vec{J}_{n}-\vec{J}_{m} \times \vec{J}_{n+N}-\vec{J}_{m+N} \times \vec{J}_{n}+\vec{J}_{m+N} \times \vec{J}_{n+N}\right)}{4 \Gamma_{m} \Gamma_{n}-\left(\vec{J}_{m} \cdot \vec{J}_{n}-\vec{J}_{m} \cdot \vec{J}_{n+N}-\vec{J}_{m+N} \cdot \vec{J}_{n}+\vec{J}_{m+N} \cdot \vec{J}_{n+N}\right)}$

## Chapter 4

## Billiard interactions

In this chapter, we study simple interactions and identify some fundamental ones which could be used to decompose more complex interactions with larger numbers of vortices. We start with the simplest case for two vortices which is a dipole. The only possible trajectory for a single dipole on a sphere is a great circle (figure.22.(a)) and the frequency for this motion as, in (2.42), would be

$$
\omega=\frac{\Gamma}{8 \Pi \cos \theta}
$$

To add more complexity, we start by adding another dipole to our system and we locate both of them on opposite sides of a great circle (equator), we start analyze their trajectories with different initial conditions.

For two or more dipoles, the interaction terms cause the centroid path of each to deviate from its underlying geodesic trajectory, hence we view each 'ballistic element' as a billiard (see Tabachnikov (1995)). Although billiard systems have recently been studied on surfaces of constant curvature, such as a sphere (see Gutkin, Smilansky, Gutkin (1999)), these systems typically do not have long-range interactions. Because the nominal distance $\left\|X_{\alpha}-X_{\alpha+N}\right\|$ between each of the point vortices which constitute a given dipole is no longer constant, it is useful to think of each as represented by its centroid coordinate, $J_{\alpha+N}$, and although the centers-of-vorticity, $J_{\alpha}$, of each of these billiards is no longer a conserved quantity, their sum over all the billiards making up the interaction is.


Figure.22. Schematic diagram depicting the interaction parameters between two dipoles. (a) Symmetric configuration in which the centers-of-vorticity vectors are aligned and $\gamma$ denotes the orientation with respect to the equator.; (b) Generic configuration where the centers-of-vorticity are not aligned, which requires two angles $\left(\gamma_{1}, \gamma_{2}\right)$ to specify the orientation.

## 4.1 $\mathrm{N}=2$ : Fundamental interactions

We start with the most important case of two interacting dipoles, whose equations in the dipole coordinates are given by (3.21):

$$
\begin{aligned}
& \dot{\vec{J}}_{1}=\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(-\vec{R}_{1,2}-\vec{R}_{3,4}-\vec{R}_{2,3}+\vec{R}_{1,4}\right) \\
& \dot{\vec{J}}_{2}=\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(\vec{R}_{1,2}+\vec{R}_{3,4}+\vec{R}_{2,3}-\vec{R}_{1,4}\right) \\
& \dot{\vec{J}}_{3}=\frac{1}{4 \Pi} \cdot \frac{\vec{J}_{1} \times \vec{J}_{3}}{\left(1+\frac{\left\|\vec{J}_{1}\right\|^{2}-\left\|\vec{J}_{3}\right\|^{2}}{4 \Gamma_{1}^{2}}\right)}+\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(-\vec{R}_{1,2}+\vec{R}_{3,4}+\vec{R}_{2,3}+\vec{R}_{1,4}\right) \\
& \dot{\vec{J}}_{4}=\frac{1}{4 \Pi} \cdot \frac{\vec{J}_{2} \times \vec{J}_{4}}{\left(1+\frac{\left\|\vec{J}_{2}\right\|^{2}-\left\|\vec{J}_{4}\right\|^{2}}{4 \Gamma_{2}^{2}}\right)}+\frac{\Gamma_{1} \Gamma_{2}}{4 \Pi}\left(\vec{R}_{1,2}-\vec{R}_{3,4}+\vec{R}_{2,3}+\vec{R}_{1,4}\right)
\end{aligned}
$$

Where

$$
\begin{aligned}
& \vec{R}_{1,2}=\frac{\left(\vec{J}_{1} \times \vec{J}_{2}+\vec{J}_{1} \times \vec{J}_{4}-\vec{J}_{2} \times \vec{J}_{3}+\vec{J}_{3} \times \vec{J}_{4}\right)}{4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}+\vec{J}_{1} \cdot \vec{J}_{4}+\vec{J}_{2} \cdot \vec{J}_{3}+\vec{J}_{3} \cdot \vec{J}_{4}\right)} \\
& \vec{R}_{1,4}=\frac{\left(\vec{J}_{1} \times \vec{J}_{2}-\vec{J}_{1} \times \vec{J}_{4}-\vec{J}_{2} \times \vec{J}_{3}-\vec{J}_{3} \times \vec{J}_{4}\right)}{-4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}-\vec{J}_{1} \cdot \vec{J}_{4}+\vec{J}_{2} \cdot \vec{J}_{3}-\vec{J}_{3} \cdot \vec{J}_{4}\right)} \\
& \vec{R}_{2,3}=\frac{\left(-\vec{J}_{1} \times \vec{J}_{2}-\vec{J}_{1} \times \vec{J}_{4}-\vec{J}_{2} \times \vec{J}_{3}+\vec{J}_{3} \times \vec{J}_{4}\right)}{-4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}+\vec{J}_{1} \cdot \vec{J}_{4}-\vec{J}_{2} \cdot \vec{J}_{3}-\vec{J}_{3} \cdot \vec{J}_{4}\right)} \\
& \vec{R}_{1,2}=\frac{\left(\vec{J}_{1} \times \vec{J}_{2}-\vec{J}_{1} \times \vec{J}_{4}+\vec{J}_{2} \times \vec{J}_{3}+\vec{J}_{3} \times \vec{J}_{4}\right)}{4 \Gamma_{1} \Gamma_{2}-\left(\vec{J}_{1} \cdot \vec{J}_{2}-\vec{J}_{1} \cdot \vec{J}_{4}-\vec{J}_{2} \cdot \vec{J}_{3}+\vec{J}_{3} \cdot \vec{J}_{4}\right)}
\end{aligned}
$$

The centroids are represented by the coordinates $\left(J_{1}, J_{2}\right)$, while the center-of vorticity of each is given by $\left(J_{3}, J_{4}\right)$. We use these coordinates to integrate the system using a 7 th $/ 8$ th order variable time-step Runga-Kutta method which is quite accurate for the time scales we consider and throughout the scattering phase. Our initial set-up is depicted in figure 22 where we show the two dipoles at opposite sides of the sphere with their centroids initially located at antipodal points along the equator. A special symmetric case is depicted in figure 22(a) where the dipoles are headed directly towards each other. In this case, their center-ofvorticity vectors ( $J_{1}$ and $J_{2}$ ) are aligned and the orientation of the system with respect to the equator is denoted by the angle $\gamma$. The more generic case is shown in figure 22(b) where the centers-of-vorticity vectors are not perfectly aligned, hence one requires two angles $\left(\gamma_{1}, \gamma_{2}\right)$ to fully specify the initial configuration.

### 4.1.1 Exchange scattering



Figure.23. Symmetric exchange scattering on rotating and non-rotating sphere for $\mathrm{J}=0$ equal dipoles, With these initial conditions, two exchange events take place at antipodal points during one periodic cycle.

The first and simplest example of a scattering event is called exchange scattering and is shown in figure 3. Shown is a symmetric case of exchange scattering of two equal strength
dipoles which initially have orientation $\gamma_{1}=\gamma_{2}=\pi / 2$. In this case $J_{1}+J_{2} \equiv J=0$. Two exchange events take place per cycle (i.e. the dipoles exchange partners) at antipodal points on the sphere (when there is no rotation). Figure 23(a) shows the exchange scattering on the non-rotating sphere, while figure 23(b),(c) shows trajectories on the rotating sphere where the ratio of dipole frequencies to rotation frequencies are (b) 1:1; (c) 2:1. Figures 23(d)-(f) show the same events but in the dipole coordinate system. The centroid paths are depicted here in red and green.

### 4.1.2 Non-exchange scattering

The second scattering event, called non-exchange scattering is depicted in figure 24 for two equal strength dipoles that retain their partners throughout the cycle. Figure 24(a) shows the basic interaction on the non-rotating sphere in the case where $J \neq 0$. Although there is a clear interaction between the two dipoles as indicated by the deviation of the dipoles from great circle paths, no partner exchange takes place throughout the period and the dipoles avoid direct collision. Figure 24(b) shows a long time trajectory on the rotating sphere, while figure 24(c) shows the tips of the J1 and J2 vectors during this event. Whether the orbit shown in figure 24(b) ultimately closes up, or densely covers a portion of the spherical surface crucially depends on the initial orientation of the two dipoles, and both periodic orbits and quasi-periodic orbits co-exist. Figure 24(d) show the same for the rotating sphere where the frequency ratio is $1: 1$.


Figure.24. Non-exchange scattering event between two equal strength dipoles. Each dipole retains its partner throughout the event. (a) Non-rotating sphere, $\gamma_{1}=90 \mathrm{deg}, \gamma_{2}=80 \mathrm{deg}$; (b) Long-time trajectory on the rotating sphere; (c) Tips of the J1 and J2 vectors; (d) Tips of the J3 and J4 vectors from north pole view


Figure.24, Continued. Non-exchange scattering event between two equal strength dipoles. Each dipole retains its partner throughout the event. (a) Non-rotating sphere, $\gamma_{1}=90 \mathrm{deg}, \gamma_{2}=80 \mathrm{deg}$; (b) Long-time trajectory on the rotating sphere; (c) Tips of the J1 and J2 vectors; (d) Tips of the J3 and J4 vectors from north pole view

### 4.1.3 Loop scattering

Shown in figure 25 is an example of a loop-scattering interaction (head-on) for two unequal dipoles in which the frequency ratio is $3: 1$. Figure 25 (a) shows a case with dipole strengths $\Gamma_{1}=-\Gamma_{3}=1.0, \Gamma_{2}=\Gamma_{4}=3.0$ with orientations $\gamma_{1}=\gamma_{2}=\pi / 2$ on the non-rotating sphere.


Figure 25. Loop-exchange scattering (head-on) events for unequal dipoles. The vortex trajectories are depicted as solid curves while the centers-of-vorticity are depicted as dashed curves. (a) Non-rotating sphere; (b) Rotating sphere with 3:1 frequency ratio.


Figure.26. Loop exchange scattering (head-on) events for unequal dipoles on rotating sphere: 1:1 frequency ratio

The dipoles perform a sequence of loops as they travel around the sphere, the number of loops depends on the frequency ratio of the two dipoles. One each loop, one dipole loops inside the other, which accommodates the passage by splitting around the inner loop. Within the loop, the dipoles exchange partners, forming two new dipoles comprised of vortices of opposite sign but unequal magnitude. As a result, they move along curved trajectories within the loop. Figure 25(b) shows the same interaction on the rotating sphere. Shown is a case with unequal dipoles with frequency ratio $\omega_{1}: \omega_{2}=2: 1$ and $\omega_{1}: \omega=1: 1$. While the dipole trajectories are relatively complex, the tips of the centers-of-vorticity vectors, shown as dashed curves, move on closed periodic orbits. In this case, the overall trajectory is periodic as these vectors execute closed loops. Figure 26 shows an example of a loopscattering interaction (head-on) with a 1:1 frequency ratio.We show the vortex trajectories
(figure 26(a)), the centers-of-vorticity trajectories (figure 26(b)) projected onto a plane, and the centroid trajectories (figure 26(c)) projected onto a plane.


Figure.27. Loop exchange scattering (chasing) events for unequal dipoles.


Figure.28. Loop exchange scattering (chasing) events for unequal dipoles on rotating sphere: 1:1 frequency ratio.

By contrast, a loop-scattering interaction that we call a 'chasing' mode is shown in figures 27 and 28. Figure 27(a) shows two unequal dipoles initially aligned so that one chases the other around the non-rotating sphere creating a smaller loop during the interaction process than the head-on collision. Figure 27(b) shows the dipole coordinates during the interaction. The same interaction on the rotating sphere is shown in the sequence of figures 28(a)-(c), with a 1:1 frequency ratio. The full gamut of interactions is shown in figure 29 where we depict the scattering of two dipoles as a function of the interaction angle, varying this angle in increments of 20 deg. From this, we can see that depending on the angle, we can have loop-scattering interactions, non-exchange scattering, or exchange scattering, depending on the angle at which the dipoles are initially oriented.


Figure.29. Scattering of two equal strength dipoles as a function of interaction angle. Both loop-scattering and non-exchange scattering events are seen in these sequences. (a) $\theta=0 \mathrm{deg}$; (b) $\theta=20 \mathrm{deg}$; (c) $\theta=40 \mathrm{deg}$ : (d) $\theta=60 \mathrm{deg}$ : (e) $\theta=80 \mathrm{deg}$; (f) $\theta=100 \mathrm{deg}$; (g) $\theta=120 \mathrm{deg}$; (h) $\theta=140 \mathrm{deg}$; (i) $\theta=160 \mathrm{deg}$; (j) $\theta=180$ deg.


Figure.29, Continued. Scattering of two equal strength dipoles as a function of interaction angle. Both loopscattering and non-exchange scattering events are seen in these sequences. (a) $\theta=0 \mathrm{deg}$; (b) $\theta=20 \mathrm{deg}$; (c) $\theta=$ 40 deg : (d) $\theta=60 \mathrm{deg}$ : (e) $\theta=80 \mathrm{deg}$; (f) $\theta=100 \mathrm{deg}$; (g) $\theta=120 \mathrm{deg}$; (h) $\theta=140 \mathrm{deg}$; (i) $\theta=160 \mathrm{deg}$; (j) $\theta=180 \mathrm{deg}$.

### 4.2 Three dipoles

When three or more dipoles interact, the scattering modes described earlier for the twodipole system remain central. This is because, unless the initial conditions are chosen judiciously, only two of the dipoles within the system will typically undergo a close interaction at any given time, thus the others affect the interaction only through the far-field. As in the two dipole case, a pure exchange scattering event can take place, as shown in figure $30(\mathrm{a})$, (b) on the non-rotating sphere. The three equal strength dipoles are aligned initially so that they head for the North Pole. Note that the members of each dipole pair split off near the North Pole and pair up with a member of another dipole as they head South. The dipole coordinates are shown in figure 30(b). Figure 31 (a),(b) shows an interaction of three dipoles on a non-rotating sphere that involves both an exchange event and a loop scattering event. In figure 32, we show a panel with all of the previously documented interactions between two dipoles, retained for the three dipole problem in a setting that combines them
throughout a more complex evolution. However, if all three approach each other so that they interact simultaneously, as shown in figure 23, a much more complex process occurs that cannot easily be interpreted as combinations of simpler interactions. In a long evolutionary process of multiple dipoles, these types of interactions will not be nearly as common as the simpler interactions between pairs.

(a)

(b)

Figure.30. Pure exchange scattering of three equal strength dipoles.


Figure .31. Interaction of three dipoles which includes both an exchange scattering and a loop scattering interaction.


Figure.32. Three dipole interaction panel which shows that the basic two dipole interactions are retained, although in a setting that combines them throughout the evolution.


Figure.32, Continued Three dipole interaction panel which shows that the basic two dipole interactions are retained, although in a setting that combines them throughout the evolution.

### 4.3 General case of $\mathbf{N}$ distinct dipoles with different strengths

For the general case of N dipoles, the fundamental interactions previously described between two and three dipoles occur, but in a more complex manner. For example, in figure 33. we have three sets of dipoles with different strengths $\Gamma_{1}=1, \Gamma_{2}=1.5$ and $\Gamma_{3}=2$ located on a great circle with distances which guarantees that they arrive at the south pole all at a same time. In this figure, trajectories are more complicated, but the interactions are similar to what we had before in 4.1, although their paths leading to the interactions are more complex.

For example, figure.33. exhibits all exchange-scattering (like those shown in figure.23.), non-exchange scattering (like those shown in figure.24.), and loop-scattering (like those shown in figure.25.), as well as interactions that are new to the three-dipole case.


Figure 33. Interaction process of three dipoles which includes aspects of the two dipoles interaction problem but is generally more complex. Shown are the point vortex paths on the sphere

## Bibliography

Aref H., P.K. Newton, M.A. Stremler, T. Tokieda, D.L. Vainchtein. 2003 Vortex Crystals. Advances in Applied Mechanics Vol. 39 1-79.

Bagrets A.A., D.A. Bagrets. 1997 Nonintegrability of two problems in vortex dynamics. Chaos 7(3), 368-375.

Bogomolov V.A. 1977 Dynamics of vorticity at a sphere. Fluid Dyn. 6, 863-870.
Bogomolov V.A. 1979 Two dimensional fluid dynamics on a sphere. Izv. Atmos. Oc. Phys. 15(1), 18-22.

Bogomolov V.A. 1985 On the motion of a vortex on a rotating sphere. Izv. Atmos. Oc. Phys. 21(4), 298-302.

Borisov A.V., A.E. Pavlov. 1998 Dynamics and statics of vortices on a plane and a sphere I. Regular and Chaotic Dynamics T.3. 1, 28-38.

Borisov V.A., V.G. Lebedev. 1998 Dynamics of three vortices on a plane and a sphere II. Regular and Chaotic Dynamics T.3. 99-114.

Cabral H.E., K.R. Meyer, D.S. Schmidt. 2003 Stability and bifurcations for the N+1 vortex problem on the sphere. Regular and Chaotic Dynamics, 8(3), 259-282.

Chern S.J. 1991 Mathematical Theory of the Barotropic Model in Geophysical Fluid Dynamics. Ph.D. Dissertation, Cornell University.

DiBattista M.T. 1997 Vortex pairs on the rotating sphere. Ph.D. Thesis, Columbia University.
DiBattista M.T., L.M. Polvani 1998 Barotropic vortex pairs on a rotating sphere. Journal of Fluid Mechanics 358, 107-133.

Gill A.E. 1982 Atmosphere-Ocean Dynamics, International Geophysics Series 30, Academic Press, New York.

Hobson D.D. 1991 A point vortex dipole model of an isolated modon. Phys. Fluids A 3(12), Dec. 3027-3033.

Jamaloodeen M. 2000 Hamiltonian methods for some geophysical vortex dynamics models. Ph.D. Thesis,University of Southern California.

Khushalani B. 2004 Symplectic subcluster methods and periodic vortex motion on a sphere. Ph.D. Thesis,University of Southern California.

Kidambi R. 1999 Integrable point vortex motion on a sphere. Ph.D. Thesis,University of Southern California.

Kidambi R., P.K. Newton 1998 Motion of three point vortices on a sphere. Physica D 116 143-175.

Kidambi R., P.K. Newton 1999 Collision of three vortices on a sphere. Il Nuovo Cimento 22 C(6) 779-791.

Kidambi R., P.K. Newton 2000 Streamline topologies for integrable vortex motion on a sphere. Physica D 140 95-125.

Kimura Y. 1999 Vortex motion on surfaces of constant curvature. Proc. Roy. Soc. Lond. A, 455, 245-259.

Klyatskin K.V., G.M. Reznik. 1989 Point vortices on a rotating sphere. Oceanology 29(1) 12-16.

Kurakin L.G. 2004 On nonlinear stability of the regular vortex systems on a sphere. Chaos 14(3) Sept., 592-602.

Laurent-Polz F. 2002 Etude GIeomIetrique de la Dynamique de N Tourbillons Ponctuels sur une Sphere. Ph.D. Thesis,University of Nice.

Laurent-Polz F. 2002 Point vortices on the sphere: A case of opposite vorticities. Nonlinearity 15, 143-171.

Laurent-Polz F., J. Montaldi, M. Roberts. 2004 Stability of relative equilibria of point vortices on the sphere. preprint.

Lim C.C., J. Montaldi, M.R. Roberts 2001 Relative equilibria of point vortices on the sphere. Physica D 148, 97-135.

Lim C.C, J. Nebus, S.M. Assad 2003a Monte-Carlo and polyhedron based simulations I. Extremal states of the logarithmic N-body problem on a sphere. Discrete and Cont. Dyn. Sys. B 3(3), 213-341.

Lim C.C, J. Nebus, S.M. Assad 2003b A Monte-Carlo algorithm for the free and coaxial extremal states of the vortex N-body problem on a sphere. Physica A 328, 53-96.

Marsden J.E., S. Pekarsky, S. Shkoller. 1999 Stability of relative equilibria of point vortices on a sphere and symplectic integrators. Il Nuovo Cimento 22, 793-802.

Matsuoka C., K. Nozaki. 1992 Point vortex model of a modon solution. Phys. Fluids $B 4$ (3), March 551-558.

Nebus J. 2003 Monte Carlo Simulations, Statistical Mechanics, and Ground States of the Logarithmic Potential. Ph.D. Thesis,R ensselaer Polytechnic University.

Neven E.C. Modons on a sphere. 1993 Ph.D. Thesis,Eindhoven University of Technology.

Newton P.K. 2001 The N-Vortex Problem: Analytical Techniques. Applied Math. Sci. 145, Springer-Verlag, New York.

Newton P.K., B. Khushalani. 2002 Integrable decomposition methods and ensemble averaging for non-integrable N-vortex problems. J. of Turbulence, S1468-5248.

Newton, P.K. and H. Shokraneh, The N-vortex problem on a rotating sphere I. Multifrequency configurations, Proc. Roy. Soc. Lond. A 462, 149-169, 2006.

Newton, P.K. and H. Shokraneh, The N-vortex problem on a rotating sphere IV. Dipoles as interacting billiards, to be submitted to Proc. Roy. Soc. Lond.

Newton, P.K. and H. Shokraneh, Vortex dipole coordinates on the sphere. Vortex dominated flows: A Volume Celebrating Lu Ting's 80th Birthday, chapter 10.

Pekarsky S., J.E. Marsden. 1998 Point vortices on a sphere: stability of relative equilibria. $J$. Math. Phys. 39, 5894-5907.

Polvani L.M., D.G. Dritschel. 1993 Wave and vortex dynamics on the surface of a sphere. $J$. Fluid Mech. 255, 35-64.

Pullin D., P.G. Saffman 1991 Long-time symplectic integration: the example of four-vortex motion. Proc. Roy. Soc. A 432 481-494.

Rowley C.W., J.E. Marsden. 2002 Variational integrators for degenerate Lagrangians with applications to point vortices. CDC Conference Proceedings.

Rowley C.W., J.E. Marsden, P.K. Newton. 2004 Variational integration schemes for the Nvortex problem. preprint.

Zhang M.Q., M.Z. Qin. 1993 Explicit symplectic schemes to solve vortex systems. Comp. Math. Appl. 26, 51-56.

## Appendices

## Appendix I

The following appendix contains real data analysis of streamline patterns with the goal of comparing them with those produced from the point-vortex systems on the sphere.

All data and graphs for pressure patterns have been obtained from Climate Diagnostics Center website (http://cdc.noaa.gov). There are options to plot pressure patterns, air temperature, vector winds and other weather parameters on Atmospheric Variable Plotting Page (http://www.cdc.noaa.gov/HistData/). Data coverage is from 1980 to the present and for all data there are options of averaging the last n days.

In purpose to have unique set of plots all data are related to sea level pressure and analysis level is set to be on surface. All data gathered are mean values which lead us to have daily, weakly or monthly mean value of parameters. For each specific period of time there is two set of pressure pattern, one of them is related to northern hemisphere and the other covers southern hemisphere.


Figure.I.1. One day mean sea level pressure patterns for July 2, 2003

After finding mean pressure patterns on both northern and southern hemispheres now it's time to decompose them and simplify the pressure patterns which looks complicated at first look. To start all closed patterns which do not include any other closed patterns can be considered as a center and are colored black as you can see in next figure:


Figure.I.2. Modified one day mean sea level pressure patterns for July 2, 2003

At this step the only important factor is to locate each center approximately with no concern about their size and shape. There would be several closed paths which include some center and closed patterns. We will separate each closed path with different colors.

## a) Pressure pattern graph simplification

## Coloring connected bodies

After locating centers on both hemispheres, now is time to separate closed patterns which only contain one center. Each closed pattern would lie in another closed pattern which would contain some centers and some other closed patterns. To make it simple we will use different colors to keep each closed path separate.


Figure.I.3. Connected bodies on the North Pole

## Connecting Northern and Southern Hemisphere

After separating all possible closed patterns on both northern and southern hemispheres, there would still be some uncolored regions which can not be closed in one hemisphere. In order to close these patterns we need to consider both hemispheres simultaneously and try connecting them over the full sphere. As NOAA website is using different precision for northern ham southern hemispheres, sometime It would be hard to follow a path from southern hemisphere on north hemisphere accurately, but as we are not concern about the exact shape of each closed path at this moment we have used some approximations to close all existing patterns.


Figure.I.4. Real time pressure patterns on a full sphere. To connect north pole pressure patterns to those on south pole we require 45 and 90 degree side maps of pressure patterns

## b) Kinematic decomposition

Now is time to consider how this patterns change day by day. Following changes and bifurcations of patterns on a spherical coordinate is not simple and we need to simplify these images in purpose to follow their evolutions. As it can be seen in e set of patterns, centers are continuously separating and joining together and forming new centers and in some cases similar bifurcation is happening to several sets of centers. To study this changes all centers and closed patterns can be represented in a new format like a DNA chain as follow:


Figure.I.5. Simplified one day Mean sea level pressure patterns for July 2, 2003

Now by considering a sequence of this kind of patterns it would be easier to follow how pressure patterns are bifurcating and evolving over time.

## c) Pattern sequencing

Here are simplified pressure patterns for a period of $07 / 01 / 2003$ to $07 / 04 / 2003$ :


Figure.I.6. Simplified one day Mean sea level pressure patterns for July 1, 2003


Figure.I.7. Simplified one day Mean sea level pressure patterns for July 2, 2003


Figure.I.8. Simplified one day Mean sea level pressure patterns for July 3, 2003


Figure.I.9. Simplified one day Mean sea level pressure patterns for July 4, 2003

We will consider this 4 sequential day patterns and use our simulation model to numerically solve N -vortex problem on a unit sphere and we will try different initial conditions to
generate shapes close enough to what we see in above pictures and check our model's accuracy

## Appendix II

In order to obtain similar results from our model to what we hade from real data decomposition (Appendix I) we have to study streamlines around each point vortices in any interaction.

## (a) Exchange - scattering

Following pictures exhibit streamline around each vortex while two dipoles with same strengths are heading toward each other. In this case each vortex shapes a circle shape streamline (situation $(0,0,0)$ in figure. $2(a)$.) and these circle shape streamlines remain unchanged during interactions and changing partners.


Figure.II.1. Streamline pattern bifurcations in Exchange-scattering mode


Figure.II.1, Continued. Streamline pattern bifurcations in Exchange-scattering mode

## (b) Loop - scattering

In this case two dipoles with different strengths $\left(\Gamma_{1}=1, \Gamma_{2}=3\right)$ are shooting toward each other. Again as we have dipole here, each vortex has a circle shape streamline around it and as it can be seen in following pictures this circle shape streamline remains unchanged during all interactions.


Figure.II.2. Streamline pattern bifurcations in Loop-scattering mode


Figure.II.2, Continued. Streamline pattern bifurcations in Loop-scattering mode

## (c) An example for interactions with streamline bifurcations

To observe a general case in which real bifurcations occur during interaction we start a specific case here.We put four point vortecis on a north pole, one exactly on north pole with strength $\Gamma_{1}=-1$, and other three on a circle around $\Gamma_{1}$ with equal distances from each other and strengths $\Gamma_{2}=\Gamma_{3}=\Gamma_{4}=1$.

Now we shoot a dipole of strength $\Gamma=1$, from great circle toward the North Pole. As it is clear in following pictures, several bifurcations from circle shape streamline ( case $(0,0,0)$ in figure.2(a).) to figure eight streamline (case (2,0,0) in figure.2(a)) and vice versa.


Figure.II.3. Streamline pattern bifurcations in more general case. Shooting a dipole toward North Pole while having a set of four vortices circling on the North Pole


Figure.II.3, Continued. Streamline pattern bifurcations in more general case. Shooting a dipole toward North Pole while having a set of four vortices circling on the North Pole

