A straightforward analysis leads to

$$\begin{aligned} &\lambda = a_5 - a_6 - i\beta, \quad \mu = 4a_5 - a_6 + i8\beta, \\ &\mathcal{A} = \tfrac{1}{2}, \quad \mathcal{B} = -\tfrac{1}{2}, \\ &\mathcal{C} = 0, \quad \mathcal{D} = \tfrac{3}{4(20 - i9\beta)}, \\ &\mathcal{E} = \tfrac{1}{2}D, \quad \mathcal{F} = -\tfrac{1}{12(15 - i4\beta)}. \end{aligned} \tag{34}$$

This completes the center manifold reduction.

Very interesting conclusions result, for example, with respect to the number of modes and their interplay in time, from the systematic treatment with the center manifold theory. For example, one interesting aspect is that the present codimension-two analysis can describe successive bifurcations of one unstable mode, which, in some cases can lead to chaos in time.

KARL SPATSCHEK

See also **Inertial manifolds; Invariant manifolds and sets; Synergetics**

Further Reading

Carr, J. 1981. *Applications of Center Manifold Theory*, New York: Springer

Grosche, G., Ziegler, V., Ziegler, D. & Zeidler, E. (editors). 1995. *Teubner-Taschenbuch der Mathematik, Teil II*, Stuttgart: Teubner

Guckenheimer, J. & Holmes, P. 1983. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Berlin and New York: Springer

Marsden, J.E. & McCracken, M. 1976. *The Hopf Bifurcation and Its Applications*, New York: Springer

CENTRAL LIMIT THEOREM

See **Martingales**

CHAOS VS. TURBULENCE

The notion of chaos has its genesis in the work of Henri Poincaré (*See* **Poincaré theorems**) on the three-body problem of celestial mechanics. Poincaré realized that this problem cannot be reduced to quadratures and solved in the manner of the two-body problem. A precise definition of chaos or non-integrability can be given in terms of the absence of conserved quantities necessary to yield a solution. It took several decades for the full significance of non-integrable dynamical systems to be appreciated and for the term "chaos" to be introduced (*See* **Chaotic dynamics**). An important step was the 1963 paper by Edward N. Lorenz, entitled "Deterministic Nonperiodic Flow" (Lorenz, 1963), on a model describing thermal convection in a layer of fluid heated from below. The Lorenz model truncates the basic fluid dynamical equations, written in terms of Fourier amplitudes, to just three modes (*See* **Lorenz equations**):

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz. \end{aligned} \tag{1}$$

In this system, x is the time-dependent amplitude of a stream-function mode, while y and z are mode amplitudes of the temperature field. The parameters σ, r, and b depend on the geometry, the boundary conditions, and the physical parameters of the fluid. Equations (1) are a subset of the full, infinite system of mode amplitude equations, chosen such that it exactly captures the initial instability of the thermally conducting state to convecting rolls when the parameter r, known as the Rayleigh number, is increased.

What Lorenz observed in numerical solutions of (1), and verified by analysis, was that very complicated, erratic solutions would arise when r was increased well beyond the conduction-to-convection transition. In fact, Lorenz had found the first example of what is today called a strange attractor (*See* Figure 1 and **Attractors**). System (1) is clearly deterministic, yet it can produce non-periodic solutions. There were other intriguing aspects of the solutions to (1) in the chaotic regime. Solutions arising from close initial conditions would separate exponentially in time, leading to an apparently random dependence on initial conditions of the solution after a finite time (*See* **Butterfly effect**). Today, this would be associated with the existence of a positive characteristic Lyapunov exponent. A list of "symptoms" can be established that are shared by systems having the property of chaos, including: complex temporal evolution, exponential separation from close initial conditions, a strange attractor in phase space (if the system is dissipative), and positive Lyapunov exponents. An important difference from Poincaré's work was that Lorenz's system described a dissipative system in which energy is not conserved.

From the start, the potential connection between chaos and other concepts in statistical physics, such as ergodicity and turbulence, was of central interest. For example, chaos was thought to imply ergodic

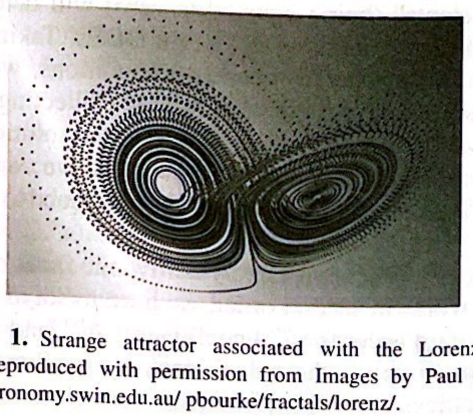

Figure 1. Strange attractor associated with the Lorenz equations. Reproduced with permission from Images by Paul Bourke, http://astronomy.swin.edu.au/ pbourke/fractals/lorenz/.

behavior in the sense of the "ergodic hypothesis" underlying equilibrium statistical mechanics (*See* **Ergodic theory**). Similarly, the connection between chaos and turbulence was sought, particularly appropriate given that Lorenz's model was of a fluid flow. Experiments on other fluid systems by Gollub, Swinney, Libchaber, and later many others established that the transition from laminar to turbulent flow typically takes place through a regime of chaotic fluid motion. The well-known route to chaos via period-doubling bifurcations of Mitchell J. Feigenbaum belongs here as well (Feigenbaum, 1980; Eckmann, 1981). In view of this, it is natural to think that turbulent flow itself is simply some kind of chaotic flow state.

Turbulence is a common state of fluid flow that shares several "symptoms" with chaotic dynamical systems, but also has distinct features not easily duplicated by chaos. The word "turbulence" was apparently first used by Leonardo da Vinci to describe a complex flow. In mathematical terms, turbulent flows should be solutions of the Navier–Stokes equation, usually written in the dimensionless form (*See* **Navier–Stokes equation**)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + R^{-1} \Delta \boldsymbol{u}, \quad (2)$$

$$\nabla \cdot \boldsymbol{u} = 0. \quad (3)$$

We have restricted attention to incompressible flows by insisting in (3) that the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ be divergence free. In (2) the field p represents the pressure—the constant density has been absorbed in the nondimensionalization. The sole dimensionless parameter R is Reynolds number. In terms of physical variables $R = UL/\nu$, where U is a typical scale of velocity, L a typical length scale of the flow, and n is the kinematic viscosity of the fluid. For small values of R, say $0 < R \leq 1$, the flow is laminar. For moderate R, say $1 < R \leq 100$, various periodic flow phenomena may arise, such as the shedding of vortices from blunt bodies. For large R, the flow eventually breaks down into many interacting eddies—this is turbulent flow. Since most flowing fluid is, in fact, flowing at large R, turbulence is the prevailing flow state of fluids in our surroundings (oceans and atmosphere), in the universe in general, in many industrial processes, and to some extent, within our bodies.

The characterization of what makes a flow turbulent is not nearly so clear as what makes a dynamical system chaotic. First, the issue of whether the particular set of nonlinear partial differential equations (2) and (3) even has a smooth solution for all time, given smooth initial conditions, is still unsettled and is one of the prize challenges set by the Clay Mathematics Institute (http://www.claymath.org). In spite of several attempts, a convincing example of a flow with smooth initial conditions, evolving under (2) and (3), that develops a singularity in a finite time has not been found. Conversely, there is no proof that solutions with the requisite number of derivatives will exist for all time.

Turbulent flows are also recognized by a variety of "symptoms." The flow velocity as a function of time at any given point in a turbulent flow is a random function (roughly a Gaussian). However, the overall nature of the velocity field viewed as a random vector field is not Gaussian. The random nature of turbulent velocity fields is today thoroughly familiar to the flying public. The randomness is not just temporal at a fixed point in space; the spatial variation of the flow field at a given time constitutes a multitude of interacting eddies of different sizes. Because of their random character, turbulent flows stir vigorously, leading to rapid dispersal of a passively advected substance or a field, such as temperature, and to a rapid exchange of momentum with contiguous fluid. In the classic pipe flow experiment of Osborne Reynolds, for example, in which the transition from laminar to turbulent flow was first demonstrated to depend only on the dimensionless number R, a streak of dye introduced at the inlet would remain a thin streak (except for a bit of molecular diffusion) when the flow in the pipe was laminar. When the flow rate was increased and the flow became turbulent, the dye rapidly dispersed across the pipe.

In a turbulent flow, the large scales of motion, which are typically in contact with some kind of forcing from the outside, will generate smaller scales through interactions and instabilities. This process continues through a broad range of length scales, ultimately reaching small scales where molecular dissipation is effective and quells the motion altogether. The repeated process of "handing down" energy from larger scales to smaller scales is a key process in turbulence. It is usually referred to as the Kolmogorov cascade (*See* **Kolmogorov cascade**). The qualitative nature of this process was already envisaged by Lewis Fry Richardson and was described by him in an adaptation of a verse by Jonathan Swift:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity (in the molecular sense).

Because of its broad range of length scales, the energy in a turbulent flow may be considered partitioned among modes of different wavenumbers k. The energy spectrum $E(k)$ is defined such that $E(k)\,\mathrm{d}k$ is the amount of kinetic energy of the turbulent flow associated with motions with wavenumbers between k and $k + \mathrm{d}k$. The cascade implies a transfer of energy from scale to scale with a characteristic energy flux per unit mass, ε, which must also be equal to the rate at which energy is fed to the flow from the largest scales, and to the rate at which energy is dissipated by viscosity at the smallest scales. A simple dimensional argument then (*See* **Dimensional**

analysis) gives the dependence of $E(k)$ on ε and k to be

$$E(k) = C\varepsilon^{2/3}k^{-5/3}. \qquad (4)$$

This is the well-known Kolmogorov spectrum, predicted by Andrei N. Kolmogorov in 1941 (Hunt et al., 1991; Frisch, 1995) and only subsequently verified by experiments in a tidal channel (see Figure 2).

Turbulence has many further intriguing statistical properties, which remain subjects of active research. A major shift in our thinking on turbulence occurred in the late 1960s and in the 1970s when experiments by Kline and Brown & Roshko demonstrated that even in turbulent shear flows at very large Reynolds number, one can identify coherent structures that organize the flow to some extent (Figure 3). Later investigations have shown that even in homogeneous, isotropic turbulence,

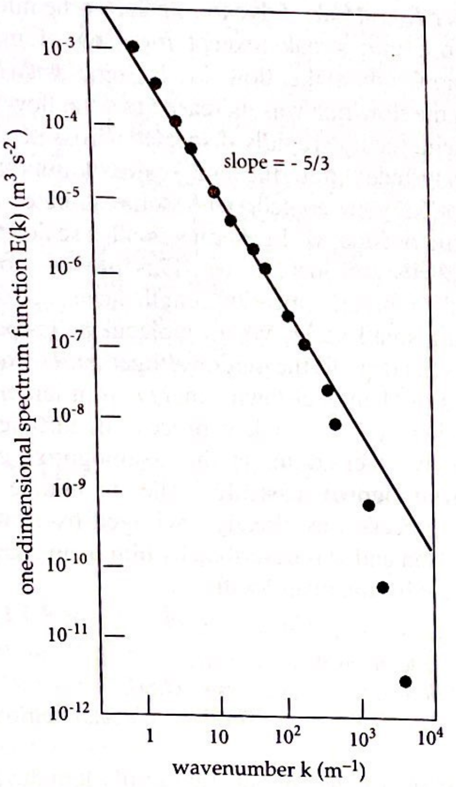

Figure 2. One-dimensional spectrum in a tidal channel from data in Grant et al. (1962).

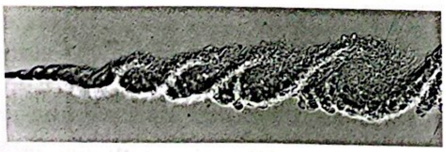

Figure 3. Coherent structures in a turbulent mixing layer. From Brown & Roshko (1974), reprinted from *An Album of Fluid Motion*, M. Van Dyke, Parabolic Press, 1982.

the flow is often organized into strong filamentary vortices. The persistence of these organized structures, which can dominate the flow for long times and interact dynamically, forces a strong coupling among the spectral modes, reducing the effective number of degrees of freedom of the problem.

Chaos and turbulence both describe states of a deterministic dynamical system in which the solutions appear random. Our current understanding of chaos is largely restricted to few-degree-of-freedom systems. Turbulence, on the other hand, is a many-degree-of-freedom phenomenon. It seems somewhat unique to fluid flows—related phenomena such as plasma turbulence or wave turbulence appear to be intrinsically different. The emergence of collective modes in the form of coherent structures in turbulence amidst the randomness is an intriguing feature, somewhat reminiscent of the mix between regular "islands" and the "chaotic sea" observed in chaotic, low-dimensional dynamical systems. The coherent structures themselves approximately form a deterministic, low-dimensional dynamical system. However, it seems impossible to fully eliminate all but a finite number of degrees of freedom in a turbulent flow—the modes not included explicitly form an essential, dissipative background, often referred to as an eddy viscosity, that must be included in the description.

Turbulence is intrinsically spatiotemporal, whereas chaotic behavior in a fluid system can be merely temporal with a simple spatial structure. It is possible for the flow field to be perfectly regular in space and time, yet the trajectories of fluid particles moving within the flow will be chaotic. This is the phenomenon of chaotic advection (*See* **Choatic advection**), which points out the hugely increased complexity of a turbulent flow relative to chaos in a dynamical system.

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See also **Attractors; Butterfly effect; Celestial mechanics; Chaotic advection; Chaotic dynamics; Diffusion; Ergodic theory; Kolmogorov cascade; Lorenz equations; Lyapunov exponents; Navier–Stokes equation; *N*-body problem; Partial differential equations, nonlinear; Period doubling; Phase space; Poincaré theorems; Routes to chaos; Shear flow; Thermal convection; Turbulence**

Further Reading

Aref, H. & Gollub, J.P. 1996. Application of dynamical systems theory to fluid mechanics. *Research Trends in Fluid Dynamics, Report of the US National Committee on Theoretical and Applied Mechanics*, edited by J.L. Lumley et al., New York: AIP Press, pp. 15–30

Eckmann, J.P. 1981. Roads to turbulence in dissipative dynamical systems. *Reviews of Modern Physics*, 53: 643–654

Feigenbaum, M.J. 1980. Transition to aperiodic behavior in turbulent systems. *Communications in Mathematical Physics*, 77: 65–86

Frisch, U. 1995. *Turbulence—The Legacy of A. N. Kolmogorov*, Cambridge and New York: Cambridge University Press
Grant, H.L., Stewart, R.W. & Moilliet, A. 1962. Turbulent spectra from a tidal channel. *J. Fluid. Mech.*, 12: 241–268
Hunt, J.C.R., Phillips, O.M., & Williams, D. (editors). 1991. Turbulence and stochastic processes: Kolmogorov's ideas 50 years on. *Proceedings of the Royal Society, London* A, 434: 1–240
Lorenz, E.N. 1963. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, 20: 130–141
Ruelle, D. 1991. *Chance and Chaos*, Princeton, NJ: Princeton University Press

CHAOTIC ADVECTION

In fluid mechanics, *advection* means the transport of material particles by a fluid flow, as when smoke from a chimney is blown by the wind. The term *passive advection* is sometimes used to emphasize that the substance being carried by the flow is sufficiently inert that it follows the flow entirely, the velocity of the advected substance at every point and every instant adjusting to that of the prevailing flow.

To describe the kinematics of a fluid, two points of view may be adopted: the Eulerian representation focuses on the velocity field $\mathbf{u}$ as a function of position and time, $\mathbf{u}(\mathbf{x}, t)$; the Lagrangian representation emphasizes the trajectories $\mathbf{x}_P(t)$ of a fluid particle as it is advected by the flow. The two points of view are linked by stating that the value of the velocity field at a given point in space and instant in time equals the velocity of the fluid element passing through that same point at that instant, that is,

$$\dot{\mathbf{x}}_P(t) = \mathbf{u}(\mathbf{x}_P(t), t). \quad (1)$$

The Eulerian representation is used extensively for measurements and numerical simulations of fluid flow since it allows one to fix the points in space and time where the field is to be determined. The Lagrangian representation, on the other hand, is often more natural for theoretical analysis, as it explicitly addresses the nonlinearity of the Navier–Stokes equation.

For a given flow, the equations of motion (1), sometimes called the *advection equations*, are a system of ordinary differential equations that define a dynamical system. These equations can be integrable or non-integrable. Chaotic advection appears when the equations are non-integrable and the trajectories of fluid elements become chaotic. The dynamical system defined by (1) has two or more degrees of freedom. For a two-dimensional time-independent or steady flow, there are just two degrees of freedom and no chaotic motion is possible. However, already for a 2-d time-dependent or a 3-d steady flow, there are enough degrees of freedom to allow for chaotic trajectories. In other words, chaotic advection can appear even for flows that would otherwise be considered laminar.

The phenomenon of chaotic advection is also known as Lagrangian chaos, or sometimes Lagrangian turbulence. Usually, the word turbulence refers to the Eulerian representation and to flows in which the velocity field fluctuates across a wide range of spatial and temporal scales with limited correlations. In such flows, the trajectories of fluid elements are always chaotic. By contrast, chaotic advection or Lagrangian chaos can arise in situations where the velocity field is spatially coherent and the time dependence is no more complicated than a simple periodic modulation.

Many examples have now been given to illustrate the point that the complexity of the spatial structure of material advected by a flow can be much greater than one might surmise from a picture of the instantaneous streamlines of the flow. Thus, in the paper that introduced the notion of chaotic advection (Aref (1984) and Figure 1), the case of two stirrers that act alternately on fluid confined to a disk was considered. Each stirrer was modeled as a point vortex that could be switched on and off. There are several parameters in the system, such as the strengths and positions of the vortex stirrer and the time interval over which each acts. For a wide range of parameter values, the dynamics is as shown in Figure 1; after just a few periods, the 10,000 particles being advected are spread out over a large fraction of the disk.

Chaotic advection gives rise to very efficient stirring of a fluid. Material lines are stretched at a rate given by the Lyapunov exponent. In bounded flows, these exponentially growing material lines have to be folded back over and over again, giving rise to ever finer and denser striations. They are familiar from the mixing of paint or from marbelized paper. On the smallest scales diffusion, takes over and smoothes the steep gradients, giving rise to mixing on the molecular scale. The interplay between stirring and diffusion is the

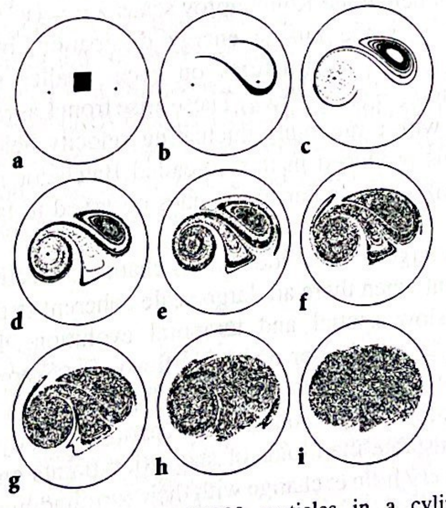

Figure 1. Spreading of 10,000 particles in a cylindrical container (disk) under the alternating action of two stirrers. The positions of the stirrers are marked by crosses. (a) initial distribution; (b)–(g) positions of the particles after 1, 2, ..., 6 periods; (h) after 9 periods; (i) after 12 periods. From (Aref, 1984).