In the form stated in Equation (1), Bernoulli's equation applies only to steady, constant-density, irrotational flow, that is, to a flow pattern that is unchanging in time and that has no vorticity.

More refined versions may be derived. Thus, in a steady, constant-density flow with vorticity, Equation (1) still holds along each streamline, but the "constant" on the right-hand side may vary from streamline to streamline. Indeed, the gradient of this changing "Bernoulli constant," ∇C, equals the Lamb vector, the vector product of flow velocity and vorticity,

$$V \times \omega = \nabla C.$$

If the flow is irrotational but unsteady, a version of Bernoulli's equation again holds, but the constant on the right-hand side of (1) is replaced by (minus) the time derivative of the velocity potential. (In an irrotational flow, the velocity field is the gradient of a scalar known as the velocity potential.) With $V = -\nabla\phi$, where ϕ is the velocity potential, we obtain Bernoulli's equation in the form

$$(\nabla\phi)^2/2 + p/\rho + gz = -\frac{\partial\phi}{\partial t}, \quad (2)$$

which, coupled with the condition of irrotational flow,

$$\Delta\phi = 0, \quad (3)$$

gives a system of two partial differential equations for the fields p and ϕ.

Bernoulli's equation in the simplistic form "high flow speed implies low pressure and vice versa" is often applied as a first, crude explanation of many flow phenomena from the ability to balance a ball atop a plume of air to the lift on an airfoil in flight. Some of these explanations are too simplistic, not to say incorrect. Nevertheless, Bernoulli's equation, when properly applied under the assumptions that ensure its validity, can be an extremely useful and powerful tool of fluid flow analysis.

It is remarkable—and important to note—that Bernoulli's equation (1) is not invariant to a Galilean transformation, ordinarily a prerequisite for a physical law to be useful. Thus, if one wants to use Bernoulli's equation (1) to calculate the pressure distribution for flow around an object, assuming the velocity field is known, it is essential to do so in a frame of reference in which the flow satisfies the necessary assumptions, in particular, that the flow is steady. The correct result is obtained by carrying out such a calculation in a frame of reference moving with the body. If the calculation is attempted in the "laboratory frame" through which the object is moving, one has to tackle the much more complex version of Bernoulli's equation given in (2). If the version in Equation (1) is applied, one obtains an incorrect result.

HASSAN AREF

See also **Fluid dynamics**

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BERRY'S PHASE

Consider the parallel transport of an orthonormal frame along a line of constant latitude on the surface of a sphere. In going once around the sphere, the frame undergoes a rotation through an angle $\Delta\theta = 2\pi \cos\alpha$, where α is the colatitude. This may be shown using the geometry of Figure 1. As is also evident from the figure, this phase shift is purely geometric in character—it is independent of the time it takes to traverse the closed loop.

This construction underlies the well-known phase shift exhibited by the Foucault pendulum as the Earth rotates through one full period. Although arising through a dynamical process involving two widely separated time scales (the period of the Earth's rotation and the oscillation period of the pendulum), the phase shift in this and other examples is now understood in a more unified way. *Holonomic* effects such as these arise in a host of applications ranging from problems in superconductivity theory, fiber optic design, magnetic resonance imaging (MRI), amoeba propulsion and robotic locomotion and control, micromoter design, molecular dynamics, rigid-body motion, vortex dynamics in incompressible fluid flows (Newton, 2001), and satellite orientation control. For a survey and further references on the use of phases in locomotion problems, see Marsden & Ostrowski (1998).

That the falling cat learns quickly to re-orient itself optimally in mid-flight while maintaining zero angular momentum is a manifestation of the fact that controlling and manipulating a system's internal or shape variables can lead to phase changes in the external, or group variables, a process that can be exploited and has deeper connections to problems related to the dynamics of Yang–Mills particles moving in their associated gauge field, a link that is the *falling cat theorem* of Montgomery (1991a) (see further discussion and references in Marsden (1992) and Marsden & Ratiu (1999)). One can read many of the original articles leading to our current understanding of the geometric phase in the collection edited by Shapere & Wilczek (1989).

Problems of this type have a long and complex history dating back to work on the circular polarization of light in an inhomogeneous medium by Vladimirskii and Rytov in the 1930s and by Pancharatnam in the 1950s, who studied interference patterns produced by plates of

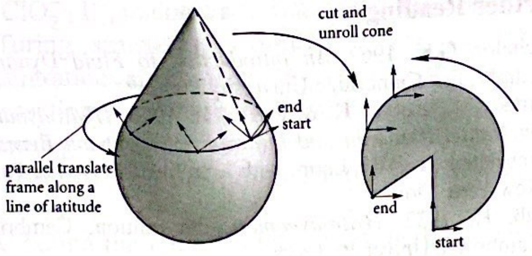

Figure 1. Parallel transport of a frame around a line of latitude.

an anisotropic crystal. Much of this early history is described in the articles by Michael Berry (Berry, 1988, 1990). The more recent literature was initiated by his earlier articles (Berry, 1984, 1985), which investigated the evolution of quantum systems whose Hamiltonian depends on external parameters that are slowly varied around a closed loop. The adiabatic theorem of quantum mechanics states that for infinitely slow changes of the parameters, the evolution of the complex wave function, governed by the time-dependent Schrödinger equation, is instantaneously in an eigenstate of the *frozen* Hamiltonian. At the end of one cycle, when the parameters recur, the wave function returns to its original eigenstate, but with a phase change that is related to the geometric properties of the closed loop. This phase change now goes by the name *Berry's phase.* Geometric developments started with the work of Simon (1983), and Marsden et al. (1989). One can introduce a bundle of eigenstates of the slowly varying Hamiltonian, as well as a natural *connection* on it; the Berry phase is then the bundle holonomy associated with this connection, while the curvature of the connection, when integrated over a closed two-dimensional (2-d) surface in parameter space gives rise to the first Chern class characterizing the topological twisting of this bundle.

The classical counterpart to Berry's phase was originally developed by Hannay (1985) (hence the terminology *Hannay's angle*) and is most naturally described by considering slowly varying integrable Hamiltonian systems in action-angle form. If we let $(I_1, \ldots, I_n; \theta_1, \ldots \theta_n)$ represent the action-angle variables of a given integrable system, then the governing Hamiltonian can be expressed as $\mathcal{H}(I_1, \ldots, I_n; R(t))$, where $R(t)$ is a slowly varying parameter that cycles through a closed loop in time period T, that is, $R(t+T) = R(t)$, $\dot{R}(t) \sim \varepsilon R$, $\varepsilon \ll 1$. The configuration space for the system is an n-dimensional torus $\mathbb{T}^n$ and we seek a formula for the angle variables as the parameter or parameters slowly evolve around the closed loop C in parameter space. The time-dependent system is governed by

$$\dot{\mathbf{I}} = \dot{R}(t) \cdot \frac{\partial \mathbf{I}}{\partial R}, \tag{1}$$

$$\dot{\theta} = \omega(\mathbf{I}) + \dot{R}(t) \cdot \frac{\partial \theta}{\partial R}, \tag{2}$$

where

$$\omega(\mathbf{I}) \equiv \frac{\partial \mathcal{H}}{\partial \mathbf{I}}.$$

Since R is slowly varying, we can average the system around level curves of the frozen (i.e., $\varepsilon = 0$) Hamiltonian. If we let $\langle \quad \rangle$ denote this phase-space average, then the averaged canonical system becomes

$$\dot{\mathbf{I}} = \dot{R}(t) \cdot \left\langle \frac{\partial \mathbf{I}}{\partial R} \right\rangle \tag{3}$$

$$\dot{\theta} = \omega(\mathbf{I}) + \dot{R}(t) \cdot \left\langle \frac{\partial \theta}{\partial R} \right\rangle. \tag{4}$$

The well-known adiabatic theorem of quantum mechanics guarantees that the action variable is nearly constant due to its adiabatic invariance, whereas the angle variables can be integrated over period T

$$\theta_T^i = \int_0^T \omega^i(\mathbf{I})\mathrm{d}t + \int_0^T \dot{R}(t) \cdot \left\langle \frac{\partial \theta^i}{\partial R} \right\rangle \mathrm{d}t \tag{5}$$

$$= \theta_{\mathrm{d}} + \theta_{\mathrm{g}}. \tag{6}$$

The first term, θ_{d}, called the dynamic phase is due to the frozen system, while the second term, θ_{g}, arises from the time variation. This geometric phase can be rewritten in a revealing manner as

$$\theta_{\mathrm{g}} = \int_0^T \dot{R}(t) \cdot \left\langle \frac{\partial \theta^i}{\partial R} \right\rangle \mathrm{d}t \tag{7}$$

$$= \oint \left\langle \frac{\partial \theta^i}{\partial R} \right\rangle \mathrm{d}R. \tag{8}$$

The contour integral is taken over the closed loop C in parameter space. Although arising through a dynamical process, it is ultimately a purely geometric quantity that results from a delicate balance of two compensating effects in the limit $\varepsilon \to 0$. On the one hand, $T \to \infty$ in (7), while on the other, $\dot{R}(t) \to 0$. Their rates exactly balance so that the integral leaves a residual term in the limit $\varepsilon = 0$, as given in (8).

A nice example developed in Hannay (1985) is the *bead-on-hoop* problem in which a frictionless bead is constrained to slide along a closed planar wire hoop that encloses area $\mathcal{A}$ and has perimeter length $\mathcal{L}$. As the bead slides around the hoop, the hoop is slowly rotated about its vertical axis (which is aligned with the gravitational vector) through one full revolution. We are interested in the angular position of the bead with respect to a fixed point on the hoop after one full revolution of the hoop. When compared with its angular position had the hoop been held fixed (the frozen problem), this angle difference would represent the geometric phase and is given by

$$\Delta\theta = -8\pi^2 \mathcal{A}/\mathcal{L}^2. \tag{9}$$

Montgomery (1991b) shows that modulo 2π, we have the following *rigid-body phase formula*:

$$\Delta\theta = -\Lambda + 2ET/R. \tag{10}$$

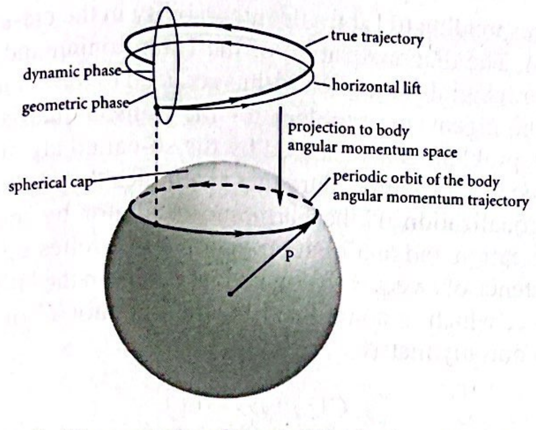

Figure 2. The geometry of the rigid-body phase formula.

Let us explain the notation in this remarkable formula. When a rigid body is freely spinning about its center of mass, one learns in mechanics that this dynamics can be described by the Euler equations, which are equations for the body angular momentum $\boldsymbol{\Pi}$. This vector in $\mathbb{R}^3$ moves on a sphere (of radius $R = \|\boldsymbol{\Pi}\|$) and describes periodic orbits (or exceptionally, heteroclinic orbits). This orbit is schematically depicted by the closed curve on the sphere shown in Figure 2. However, the full dynamics includes the dynamics of the rotation matrix for describing the attitude of the rigid body as well as its conjugate momentum. There is a projection from the full dynamic phase space (which is 6-d) to the body angular momentum space (which is 3-d). After one period of the motion on the sphere, the actual rigid-body motion was not periodic, but it had rotated about the spatial angular momentum vector by an angle $\Delta\theta$, the left-hand side of the above formula. The quantity Λ is the spherical angle subtended by the cap shown in the figure, E is the energy of the trajectory, and T is the period of the closed orbit on the sphere. A detailed history of this formula is given in Marsden & Ratiu (1999).

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See also **Adiabatic invariants; Averaging methods; Hamiltonian systems; Integrability; Phase space**

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BETHE ANSATZ

The Bethe ansatz is the name given to a method for exactly solving quantum many-body systems in one spatial dimension (1-d) or classical statistical lattice models (vertex models) in two spatial dimensions (Baxter, 1982; Korepin et al., 1993). The method was developed by Hans Bethe in 1931 (Bethe, 1931) in order to diagonalize the Hamiltonian of a chain of N spins with isotropic exchange interactions, introduced by Werner Heisenberg some years before as the simplest model for a 1-d magnet. This result was achieved by assuming the wave function to be of the form

$$f(x_1, x_2, \dots, x_M) = \sum_P A_P \, e^{\mathrm{i} \sum_{j=1}^{M} k_{P_j} x_j} \quad (1)$$

with the sum performed on all possible permutations P of M distinct wave numbers $\{k_1, \dots, k_M\}$, corresponding to down spins in the system (Bethe ansatz). By imposing invariance under the physical symmetries of the system (discrete translations and total spin rotations), Bethe obtained conditions on the coefficients A_P, which were satisfied if a set of M nonlinear equations (Bethe equations) in N complex parameters (Bethe numbers) were fulfilled. Surprisingly, the wave functions thus constructed were simultaneous eigenfunctions not only of the translation operator, the total spin $\boldsymbol{S}$, and its projection S_z along the z-direction but also of the isotropic Heisenberg Hamiltonian

$$H = \sum_{i=1}^{N} \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} - \frac{1}{4} \right). \quad (2)$$

The energy and the crystal momentum were expressed as symmetric functions of the Bethe numbers; thus, the eigenvalue problem for H was reduced to the solution of an algebraic problem—solution of the Bethe equations.