

The N -vortex problem on a rotating sphere.

II. Heterogeneous Platonic solid equilibria

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We describe a new method of constructing point vortex equilibria on a sphere made-up of N vortices with different strengths. Such equilibria, called *heterogeneous equilibria*, are obtained for the five Platonic solid configurations, hence for $N = 4, 6, 8, 12, 20$. The method is based on calculating a basis set for the nullspace of a matrix obtained by enforcing the necessary and sufficient condition that the mutual distances between each pair of vortices remain constant. By symmetries inherent in the Platonic solid configurations, this matrix is reduced for each case and we call the dimension of the nullspace the *degree of heterogeneity* of the structure. For the tetrahedron ($N=4$) and octahedron ($N=6$), the degree of heterogeneity is 4 and 6, respectively, hence we are free to choose each of the vortex strengths independently. For the cube ($N=8$), the degree of heterogeneity is 5, for the icosahedron ($N=12$) it is 7, while for the dodecahedron ($N=20$) it is 4. Thus, the entire set of equilibria based on the Platonic solid configurations is obtained, including substructures associated with each configuration constructed by taking different linear combinations of the basis elements.

Keywords: relative equilibria; Platonic solids; vortex crystals; N -vortex problem

1. Introduction

In this paper, we describe a new method of finding point vortex equilibria on the sphere made up of collections of vortices of different strengths. The identification of such equilibria, which we call *heterogeneous equilibria*, is rare. This is because in the traditional way of finding equilibria for point vortex systems, one first chooses the vortex strengths, $\Gamma_i \in \mathbb{R}$ ($i = 1, \dots, N$), and then one attempts to solve the N nonlinear point vortex equations for special configurations that remain rigid. The book of Newton (2001) gives the necessary background, while the review paper of Aref *et al.* (2003) contains a comprehensive account of the current state-of-the-art regarding vortex equilibrium patterns along with a fairly complete bibliography on the topic. If the patterns do not move, they are classified as *fixed equilibria*, whereas if they rotate or translate, they are *relative equilibria*. Typically, the vortex strengths are chosen so that all are equal (as in Lim *et al.* 2001; Newton & Shokraneh 2006) in which case the strengths can be scaled out of the equations. With this

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simplification, Campbell & Ziff (1978) claim to have found all linearly stable patterns in the plane for $N \leq 30$, a claim that has so far stood the test of time. Or, if they are chosen in equal and opposite pairs so that the total vorticity is zero (see Laurent-Polz 2002), the problem also simplifies considerably. In cases where the number of vortices is small, a complete characterization can also sometimes be achieved (see Kidambi & Newton (1998) for $N=3$). All of these choices considerably simplify what is more generally a complicated problem associated with characterizing solutions of *Kelvin's variational principle*, as explained in Aref *et al.* (2003). Additionally, one typically uses symmetric configurations (see Aref 1982; Lim *et al.* 2001; Newton & Shokraneh 2006) in order to make further progress, such as placing the vortices on polygonal rings, or nested rings (Aref *et al.* 2003).

There are two notable exceptions to this in the literature. The paper of Lewis & Ratiu (1996), who placed the vortices on nested polygons where the number of vertices in the polygons being nested is commensurate, is the only one we know of that identifies equilibria in the planar point vortex system with non-trivial choices for the vortex strengths. They identify collections of rotating n - and kn -gon structures that remain rigid, i.e. the mutual distances between each pair of vortices remains fixed in time. The second exception is the paper of Aref & Vainchtein (1998), in which asymmetric equilibria are *grown* from symmetric states by using a numerical continuation method in which a particle (i.e. zero strength vortex) is initially placed at a stagnation point associated with a known equilibrium pattern, then the vortex strength is used as a continuation parameter and increased as the structure is allowed to deform in such a way as to remain rigid. The goal of this study was to 'grow' the new vortex until it had the same strength as all the others, yielding an asymmetric pattern. However in the process, the rigid structures obtained this way are made-up of $N-1$ vortices of equal strength, and one vortex of a different strength. As such, they are heterogeneous equilibria, although of a somewhat special kind.

We construct the heterogeneous equilibria by a different procedure. First, we choose the geometric configuration (which initially defines all the intervortical distances $l_{ij}(0)$), and then ask what choice of Γ_i s leads to an equilibrium structure (i.e. $l_{ij}(t) = l_{ij}(0)$, $t > 0$). The procedure leads to a complete characterization of the vortex strength vector $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \dots, \Gamma_N) \in \mathbb{R}^N$ for which the given configuration remains rigid, i.e. it is a necessary and sufficient condition for the given pattern to be an equilibrium. Since the point vortex equations are linear in the strengths, Γ_i , when we enforce the $\binom{N}{2}$ conditions that each of the intervortical distances remain fixed, one obtains a linear system of $\binom{N}{2}$ equations for the N -vortex strengths, $\tilde{\mathbf{A}}\mathbf{\Gamma} = 0$, where $\tilde{\mathbf{A}}$ is a matrix with N columns and $\binom{N}{2}$ rows. Since $\binom{N}{2} > N$ for $N \geq 4$, the system is overdetermined and unless some of the equations are redundant, there are no solutions. Since redundancy of the equations is closely related to symmetries of the configurations, for $N \geq 4$, configurations with no symmetries have no equilibria, giving an indication of why equilibria for *asymmetric* structures are more difficult to obtain. By taking into account all inherent symmetries of the configuration, we reduce the matrix $\tilde{\mathbf{A}}$ to obtain the reduced matrix \mathbf{A} , and arrive at a general criterion for an equilibrium structure, namely that the vortex strength vector $\mathbf{\Gamma}$ be an element of the nullspace of \mathbf{A} , i.e. $\mathbf{\Gamma} \in \mathcal{N}(\mathbf{A})$. The method leads to non-trivial heterogeneous equilibria as long as $\mathcal{N}(\mathbf{A})$ is non-empty. We call the dimension of the nullspace, d , the degree of heterogeneity of the structure as it represents the number of independent vortex strengths that can be used to construct an

equilibrium. Since $\text{Rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = N$, we have that $d = N - \text{Rank}(\mathbf{A})$. Because the basis elements of $\mathcal{N}(\mathbf{A})$ can be chosen in arbitrary linear combinations, the equilibria associated with each Platonic solid also includes structures with one or more of the basis elements missing (i.e. if the vortex strengths are chosen to be zero). We call these *substructures* of the Platonic solid. Note that only configurations in which *all* N points remain rigid are obtained by our method, regardless of whether the vortex strengths are zero or non-zero.

We focus on the five Platonic solids: the tetrahedron ($N=4$), octahedron ($N=6$), cube ($N=8$), icosahedron ($N=12$) and dodecahedron ($N=20$), which can all be inscribed inside a sphere. A point vortex is placed at each vertex and by requiring that the intervortical distances l_{ij}^2 be constant, we obtain a complete characterization of all vortex strengths that allow the Platonic solid to remain rigid. We obtain the degree of heterogeneity of each of these structures and, in the process, identify many new examples of equilibria that have not been identified previously. The method holds great promise for the general construction of heterogeneous equilibria based on other patterns, and even asymmetric patterns both in the plane and on the sphere.

2. Equations of motion for relative equilibria

Consider the equations of motion for N -point vortices on a rotating sphere, as written by Newton & Shokraneh (2006) (Part I in this sequence),

$$\begin{aligned} \dot{\mathbf{x}}_i &= \frac{1}{4\pi} \sum_{k=1}^{N'} \Gamma_k \frac{\mathbf{x}_k \times \mathbf{x}_i}{(1 - \mathbf{x}_i \cdot \mathbf{x}_k)} + \Omega \hat{\mathbf{e}}_z \times \mathbf{x}_i \quad (i = 1, \dots, N) \\ \mathbf{x}_i &\in \mathbb{R}^3, \quad \|\mathbf{x}_i\| = 1. \end{aligned} \tag{2.1}$$

The prime on the summation indicates that the singular term $k=i$ is omitted and initially, the vortices are located at the given positions $\mathbf{x}_i(0) \in \mathbb{R}^3$ ($i = 1, \dots, N$). The denominator in (2.1) is the intervortical distance, l_{ik} , between vortex Γ_i and Γ_k since $l_{ik}^2 \equiv \|\mathbf{x}_i - \mathbf{x}_k\|^2 = 2(1 - \mathbf{x}_i \cdot \mathbf{x}_k)$.

The evolution equations for these relative distances are obtained as follows. Using equation (2.1), taking the difference of the equations for the i th and the j th vortex,

$$\begin{aligned} \dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j &= \frac{1}{2\pi} \sum_{k=1, k \neq i}^N \Gamma_k \frac{\mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} - \frac{1}{2\pi} \sum_{k=1, k \neq j}^N \Gamma_k \frac{\mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} + \Omega \hat{\mathbf{e}}_z \times (\mathbf{x}_i - \mathbf{x}_j) \\ &= \frac{1}{2\pi} \sum_{k=1, k \neq i, k \neq j}^N \Gamma_k \frac{\mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} - \frac{1}{2\pi} \sum_{k=1, k \neq j, k \neq i}^N \Gamma_k \frac{\mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} \\ &\quad + \frac{1}{2\pi} \Gamma_j \frac{\mathbf{x}_j \times \mathbf{x}_i}{l_{ij}^2} - \frac{1}{2\pi} \Gamma_i \frac{\mathbf{x}_i \times \mathbf{x}_j}{l_{ji}^2} + \Omega \hat{\mathbf{e}}_z \times (\mathbf{x}_i - \mathbf{x}_j) \\ &= \frac{1}{2\pi} \sum_{k=1, k \neq i, k \neq j}^N \Gamma_k \left[\frac{\mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} - \frac{\mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} \right] - \frac{1}{2\pi} (\Gamma_j + \Gamma_i) \frac{\mathbf{x}_i \times \mathbf{x}_j}{l_{ij}^2} \\ &\quad + \Omega \hat{\mathbf{e}}_z \times (\mathbf{x}_i - \mathbf{x}_j). \end{aligned} \tag{2.2}$$

Now, noting that

$$2(\mathbf{x}_i - \mathbf{x}_j) \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) \equiv \frac{d(l_{ij}^2)}{dt}, \quad (2.3)$$

take the dot product of equation (2.2) with $2(\mathbf{x}_i - \mathbf{x}_j)$

$$\begin{aligned} 2(\mathbf{x}_i - \mathbf{x}_j) \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) &\equiv \frac{d(l_{ij}^2)}{dt} \\ &= \frac{1}{\pi} (\mathbf{x}_i - \mathbf{x}_j) \cdot \sum_{k=1, k \neq i, k \neq j}^N \Gamma_k \left[\frac{\mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} - \frac{\mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} \right] \\ &\quad - \frac{1}{\pi} (\Gamma_j + \Gamma_i) (\mathbf{x}_i - \mathbf{x}_j) \cdot \frac{(\mathbf{x}_i \times \mathbf{x}_j)}{l_{ij}^2} \\ &\quad + 2\Omega (\mathbf{x}_i - \mathbf{x}_j) \cdot \hat{\mathbf{e}}_z \times (\mathbf{x}_i - \mathbf{x}_j). \end{aligned} \quad (2.4)$$

The last two terms in equation (2.4) are both identically zero, hence we are left with

$$\begin{aligned} \frac{d(l_{ij}^2)}{dt} &= \frac{1}{\pi} (\mathbf{x}_i - \mathbf{x}_j) \cdot \sum_{k=1, k \neq i, k \neq j}^N \Gamma_k \left[\frac{\mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} - \frac{\mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} \right] \\ &= \frac{1}{\pi} \sum_{k=1, k \neq i, k \neq j}^N \Gamma_k \left[\frac{\mathbf{x}_i \cdot \mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} - \frac{\mathbf{x}_i \cdot \mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} - \frac{\mathbf{x}_j \cdot \mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} + \frac{\mathbf{x}_j \cdot \mathbf{x}_k \times \mathbf{x}_j}{l_{jk}^2} \right]. \end{aligned}$$

The first and fourth terms in this sum are identically zero yielding

$$\pi \frac{d(l_{ij}^2)}{dt} = \sum_{k=1, k \neq i, k \neq j}^N \Gamma_k \left[\frac{\mathbf{x}_j \cdot \mathbf{x}_k \times \mathbf{x}_i}{l_{jk}^2} - \frac{\mathbf{x}_j \cdot \mathbf{x}_k \times \mathbf{x}_i}{l_{ik}^2} \right] = \sum_k'' \Gamma_k V_{ijk} d_{ijk},$$

where $d_{ijk} \equiv [(1/l_{jk}^2) - (1/l_{ik}^2)]$. Here, the '' means the summation excludes $k=i$ and j . V_{ijk} is the volume of the parallelepiped formed by the vectors $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$,

$$V_{ijk} = \mathbf{x}_i \cdot (\mathbf{x}_j \times \mathbf{x}_k) \equiv \mathbf{x}_j \cdot (\mathbf{x}_k \times \mathbf{x}_i) \equiv \mathbf{x}_k \cdot (\mathbf{x}_i \times \mathbf{x}_j).$$

Notice that the sign of V_{ijk} can be positive or negative depending on whether the vectors form a right- or left-handed coordinate system. The relative equations of motion yield immediately necessary and sufficient conditions for relative equilibria,

$$\frac{dl_{ij}^2}{dt} = 0, \quad \forall i, j, 1, \dots, N, \quad i \neq j,$$

for all $\binom{N}{2}$ distances between any two vortices. This equation does not distinguish between fixed or relative equilibria, hence the latter case does not yield information on the rotational frequencies. However, in Newton & Shokraneh (2006) (Part I), we obtain general formulae for these frequencies about the centre-of-vorticity axis. The analysis of the relative equilibria of the five Platonic solids simplifies because of the symmetrical nature of the configurations. These simplifications take four basic forms:

- (i) when the faces of the polyhedron are triangles, one does not need to look at the equations for all $\binom{N}{2}$ intervortical distances, but instead only the equations for the edges. Constant edges on the triangular faces are sufficient to ensure rigidity;
- (ii) when any of the terms V_{ijk} are zero, corresponding to three vortices i, j, k lying on a great circle;
- (iii) when any of the terms d_{ijk} are zero, corresponding to $l_{jk} = l_{ki}$; and
- (iv) when any of the terms d_{ijk} are common as, for instance, are the edge lengths on a regular polyhedron.

These simplifications will be exploited in varying degrees for each of the Platonic solid configurations and their analysis. We are now in a position to state a proposition regarding the *existence* and *non-existence* of equilibria.

Proposition 2.1. *A necessary and sufficient condition for an N -vortex equilibrium configuration is that*

$$\sum_k^n \Gamma_k V_{ijk} d_{ijk} = 0 \quad (i = 1, \dots, N; j = 1, \dots, N; k = 1, \dots, N; i \neq j \neq k), \quad (2.5)$$

which is a linear system of $\binom{N}{2}$ equations for the N unknowns $\Gamma_k (k = 1, \dots, N)$.

Remark. Note that (2.5) can be written as a matrix system of the form $\tilde{\mathbf{A}}\mathbf{\Gamma} = 0$, where $\mathbf{\Gamma} \equiv (\Gamma_1, \dots, \Gamma_N)^T \in \mathbf{R}^N$ and $\tilde{\mathbf{A}}$ has $\binom{N}{2}$ rows and N columns. This gives:

Corollary 2.2. *Non-trivial equilibria exist iff $\det(\tilde{\mathbf{A}}^T \tilde{\mathbf{A}}) = 0$. When this condition holds, the nullspace of $\tilde{\mathbf{A}}$, denoted $\mathcal{N}(\tilde{\mathbf{A}})$, is non-empty, and equilibria exist for all $\mathbf{\Gamma} \in \mathcal{N}(\tilde{\mathbf{A}})$.*

Definition. The dimension of $\mathcal{N}(\tilde{\mathbf{A}}) \equiv d$, where $d = N - \text{Rank}(\tilde{\mathbf{A}})$, is called the *degree of heterogeneity* of the structure.

We now use this to characterize all of the Platonic solid equilibria.

3. The tetrahedron

We begin by observing that on the tetrahedron, every l_{ij} corresponds to an edge, as shown in figure 1. Moreover, the edges have common length $l_{ij} = d$. This means that all terms in the sum (2.5) are zero since (see figure 1)

$$\left[\frac{1}{l_{jk}^2} - \frac{1}{l_{ki}^2} \right] = \left[\frac{1}{d^2} - \frac{1}{d^2} \right] = 0.$$

Hence, the matrix $\tilde{\mathbf{A}} \equiv 0$ and its nullspace has dimension four. A basis for this nullspace is the standard basis in \mathbf{R}^4 . This simple observation leads us to conclude that *any configuration of vortices on a tetrahedron is a relative equilibrium on the sphere*. The important result here is that the four vortex strengths can be chosen independently and in any linear combination, thus, each of the four configurations shown in figure 2 can be thought of as a basis element making up all possible tetrahedral equilibria. We summarize this result in the following proposition.

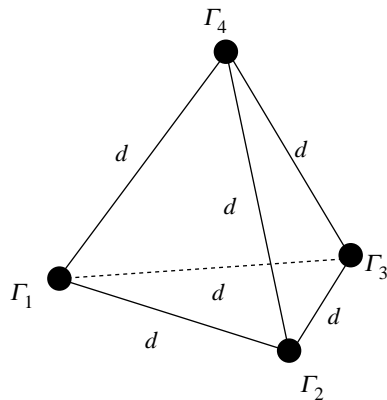


Figure 1. The tetrahedron with $N=4$ and common edge length $l_{ij} = d$.

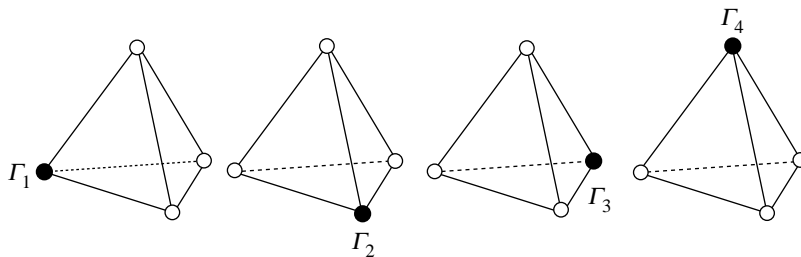


Figure 2. The four basis elements of the tetrahedron. Any linear combination of these elements forms an equilibrium. Open circles represent a vortex of zero strength.

Proposition 3.1. *Any configuration of four vortices (regardless of vortex strengths) on a tetrahedron is necessarily a relative equilibrium configuration on the sphere.*

Notice that vortices can be of arbitrary strength and, in particular, their strengths can be zero. If one or more of the vortices has strength zero, such a tetrahedral configuration is termed as a *substructure* of the ‘full’ four-vortex tetrahedral configuration. In view of this, the following corollary is immediate.

Corollary 3.2. *Any substructure of one, two or three vortices (regardless of vortex strengths) on a tetrahedron is necessarily a relative equilibrium configuration on the sphere.*

This corollary in itself is not a new result, but ties in nicely with previous results. It is known, for instance, that any two vortices on a sphere are necessarily in relative equilibrium (Kidambi & Newton 1998; Newton 2001). Likewise, any three vortices on a tetrahedron necessarily lie on an equilateral triangle, and equilateral triangle configurations have been shown to be relative equilibria (Kidambi & Newton 1998; Newton 2001).

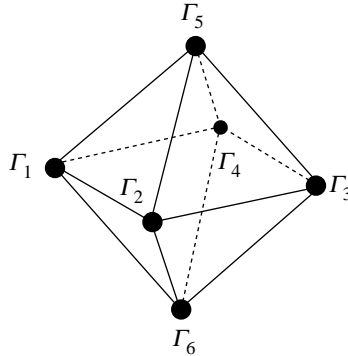


Figure 3. Vortex assignment for the octahedron with common edge $l_{ij} = d$ and vortex coordinates $\mathbf{x}_1 = (0, -1, 0)$, $\mathbf{x}_2 = (1, 0, 0)$, $\mathbf{x}_3 = (0, 1, 0)$, $\mathbf{x}_4 = (-1, 0, 0)$, $\mathbf{x}_5 = (0, 0, 1)$, $\mathbf{x}_6 = (0, 0, -1)$.

4. The octahedron

The faces of the octahedron, like the tetrahedron, are triangles. This means only the edges of the triangular faces need be considered. Consider, for example, the evolution of the edge l_{12} with reference to figure 3,

$$\pi \frac{dl_{12}^2}{dt} = \Gamma_3 V_{123} d_{123} + \Gamma_4 V_{124} d_{124} + \Gamma_5 V_{125} d_{125} + \Gamma_6 V_{126} d_{126}. \tag{4.1}$$

Each of the terms is seen to be zero as follows:

- (i) $V_{123} = 0$, as $\Gamma_1, \Gamma_2, \Gamma_3$ lie on a great circle;
- (ii) $V_{124} = 0$, as $\Gamma_1, \Gamma_2, \Gamma_4$ lie on a great circle;
- (iii) $d_{125} = [(1/l_{15}^2) - (1/l_{25}^2)] = 0$, as $l_{15} = l_{25}$; and
- (iv) $d_{126} = [(1/l_{16}^2) - (1/l_{26}^2)] = 0$, as $l_{16} = l_{26}$.

It is straightforward to show that $\Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma_1, \Gamma_2, \Gamma_4$ lie on great circles.

By cycling through all indices i, j corresponding to edges, we find the same reasoning holds for each l_{ij} . Hence, as for the tetrahedron, the matrix $\tilde{\mathbf{A}} \equiv 0$ and its nullspace has dimension six. A basis for this nullspace is the standard basis in \mathbf{R}^6 . We summarize this result in the following proposition.

Proposition 4.1. *Any configuration of six vortices (regardless of vortex strengths) on an octahedron is necessarily a relative equilibrium configuration on the sphere.*

The following corollary is also immediate.

Corollary 4.2. *Any substructure of one, two, three, four or five vortices (regardless of vortex strengths) on an octahedron is necessarily a relative equilibrium configuration on the sphere.*

Figure 4 shows the six basis elements that can be chosen in any linear combination to make-up an octahedral equilibrium, some of which have been found before by *ad hoc* methods, some of which have not previously been identified as such.

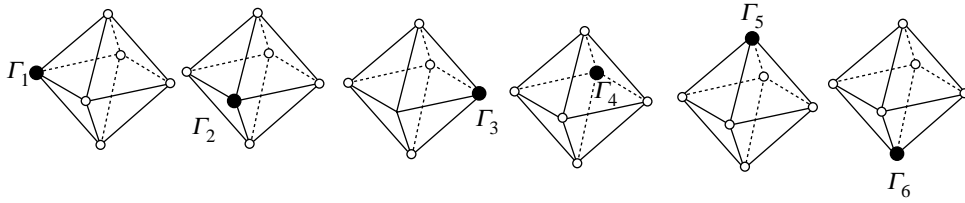


Figure 4. The six basis elements of the octahedron. Any linear combination of these forms an equilibrium. Open circles represent a vortex of zero strength.

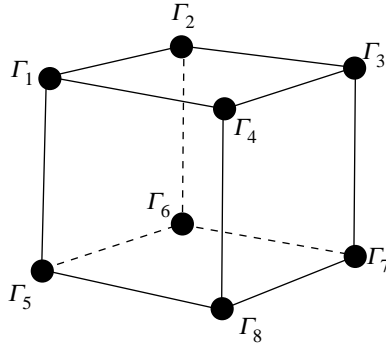


Figure 5. Vortex assignment for the cube with common edge $l_{ij} = 2/\sqrt{3}$ and vortex locations $\mathbf{x}_1 = (1/\sqrt{3})(-1, -1, 1)$, $\mathbf{x}_2 = (1/\sqrt{3})(-1, 1, 1)$, $\mathbf{x}_3 = (1/\sqrt{3})(1, 1, 1)$, $\mathbf{x}_4 = (1/\sqrt{3})(1, -1, 1)$, $\mathbf{x}_5 = (1/\sqrt{3})(-1, -1, -1)$, $\mathbf{x}_6 = (1/\sqrt{3})(-1, 1, -1)$, $\mathbf{x}_7 = (1/\sqrt{3})(1, 1, -1)$, $\mathbf{x}_8 = (1/\sqrt{3})(1, -1, -1)$.

5. The cube

The faces of the cube are not triangles, so they must be tessellated by also considering the diagonals on them. Consider the evolution of the diagonal l_{13} with reference to figure 5,

$$\begin{aligned} \pi \frac{dl_{13}^2}{dt} &= \Gamma_2 V_{132} d_{132} + \Gamma_4 V_{134} d_{134} + \Gamma_6 V_{136} d_{136} + \Gamma_8 V_{138} d_{138} \\ &+ \Gamma_5 V_{135} d_{135} + \Gamma_7 V_{137} d_{137}. \end{aligned} \tag{5.1}$$

Each of the terms is seen to be zero as follows:

- (i) $d_{132} = [(1/l_{12}^2) - (1/l_{32}^2)] = 0$, as $l_{12} = l_{32}$;
- (ii) $d_{134} = [(1/l_{14}^2) - (1/l_{34}^2)] = 0$, as $l_{14} = l_{34}$;
- (iii) $d_{136} = [(1/l_{16}^2) - (1/l_{36}^2)] = 0$, as $l_{16} = l_{36}$;
- (iv) $d_{138} = [(1/l_{18}^2) - (1/l_{38}^2)] = 0$, as $l_{18} = l_{38}$;
- (v) $V_{135} = 0$, as $\Gamma_1, \Gamma_3, \Gamma_5$ lie on a great circle; and
- (vi) $V_{137} = 0$, as $\Gamma_1, \Gamma_3, \Gamma_7$ lie on a great circle.

Since it is straightforward to prove that V_{135} and V_{137} are both zero, i.e that $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_5$ and $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_7$ lie on great circles, we do not include the proof. By symmetries, the same is true for all other diagonals.

Thus, it suffices to find conditions on the cube edges to remain fixed. Consider the equation for the edge l_{12} ,

$$\pi \frac{dl_{12}^2}{dt} = \sum_k'' \Gamma_k V_{12k} d_{12k} = \sum_k'' \Gamma_k [\mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_2)] \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right]. \tag{5.2}$$

We simplify each of the terms by noting that

$$\mathbf{x}_1 \times \mathbf{x}_2 = \begin{vmatrix} i & j & k \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{vmatrix} = \begin{bmatrix} -2/3 \\ 0 \\ -2/3 \end{bmatrix}.$$

Hence, each term in the summation in (5.2) is

$$\begin{aligned} \Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_2) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right] &= \frac{1}{3} \Gamma_k \mathbf{x}_k \cdot \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right] \\ &= \frac{4\sqrt{3}}{9} (-1)^{k+1} \alpha \Gamma_k \quad (k = 3, 4, 5, 6), \end{aligned} \tag{5.3}$$

$$\Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_2) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right] = 0 \quad (k = 7, 8). \tag{5.4}$$

The simplifications mentioned in §1 are apparent in the common term α , which arises as a result of the symmetry of the cube. The term is common to all non-zero or non-great-circle terms with $[(1/l_{jk}^2) - (1/l_{ik}^2)]$ appearing in the dynamical equation for the edges of the cube. With reference to figure 5, it is clear that

$$\alpha = \left| \frac{1}{l_{13}^2} - \frac{1}{l_{23}^2} \right| = \left| \frac{3}{(2\sqrt{2})^2} - \frac{3}{(2)^2} \right| = \frac{3}{8}.$$

Putting together equations (5.3) and (5.4) simplifies the equation for the edge l_{12} ,

$$\pi \frac{dl_{12}^2}{dt} = \frac{4\sqrt{3}}{9} \alpha \Gamma_3 - \frac{4\sqrt{3}}{9} \alpha \Gamma_4 + \frac{4\sqrt{3}}{9} \alpha \Gamma_5 - \frac{4\sqrt{3}}{9} \alpha \Gamma_6.$$

So, a necessary condition for a relative equilibrium is

$$\pi \frac{dl_{12}^2}{dt} = 0 \Rightarrow \Gamma_3 - \Gamma_4 + \Gamma_5 - \Gamma_6 = 0. \tag{5.5}$$

The analysis can be repeated for all 12 edges on the cube simply by permuting the edges into the edge l_{12} and obtaining the relevant linear equations on the vortex strengths corresponding to the given edge as in equation (5.5). The results are displayed in table 1. Some of the relations in table 1 are redundant; for example, the relations arising from l_{12} and l_{87} . When all of these redundancies are

Table 1. Linear constraints on the vortex strengths Γ_i arising from requiring that the cube edges remain constant, i.e. $l_{ij}(t) = \text{const.}$

permutation (σ)	edge	constraint on vortex strengths
(1, 2, 3, 4, 5, 6, 7, 8)	l_{12}	$-\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma_6 = 0$
(8, 7, 6, 5, 4, 3, 2, 1)	l_{87}	$-\Gamma_6 + \Gamma_5 - \Gamma_4 + \Gamma_3 = 0$
(5, 6, 2, 1, 8, 7, 3, 4)	l_{56}	$-\Gamma_2 + \Gamma_1 - \Gamma_8 + \Gamma_7 = 0$
(4, 3, 7, 8, 1, 2, 6, 5)	l_{43}	$-\Gamma_7 + \Gamma_8 - \Gamma_1 + \Gamma_2 = 0$
(4, 1, 2, 3, 8, 5, 6, 7)	l_{41}	$-\Gamma_2 + \Gamma_3 - \Gamma_8 + \Gamma_5 = 0$
(2, 3, 4, 1, 6, 7, 8, 5)	l_{23}	$-\Gamma_4 + \Gamma_1 - \Gamma_6 + \Gamma_7 = 0$
(2, 6, 7, 3, 1, 5, 8, 4)	l_{26}	$-\Gamma_7 + \Gamma_3 - \Gamma_1 + \Gamma_5 = 0$
(5, 1, 4, 8, 6, 2, 3, 7)	l_{51}	$-\Gamma_4 + \Gamma_8 - \Gamma_6 + \Gamma_2 = 0$
(6, 7, 3, 2, 5, 8, 4, 1)	l_{67}	$-\Gamma_3 + \Gamma_2 - \Gamma_5 + \Gamma_8 = 0$
(8, 5, 1, 4, 7, 6, 2, 3)	l_{85}	$-\Gamma_1 + \Gamma_4 - \Gamma_7 + \Gamma_6 = 0$
(3, 7, 2, 6, 8, 4, 1, 5)	l_{37}	$-\Gamma_2 + \Gamma_6 - \Gamma_8 + \Gamma_4 = 0$
(8, 4, 3, 7, 5, 1, 2, 6)	l_{84}	$-\Gamma_3 + \Gamma_7 - \Gamma_5 + \Gamma_1 = 0$

removed, the reduced linear system of equations is

$$\mathbf{A}\mathbf{\Gamma} = \begin{bmatrix} 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \end{bmatrix} = 0. \tag{5.6}$$

The structure of \mathbf{A} yields immediately the solution $\mathbf{\Gamma} = (1, 1, 1, 1, 1, 1, 1, 1)^T$, the case of all identical vortices. All relative equilibrium solutions are found by finding $\mathcal{N}(\mathbf{A})$, the nullspace (kernel) of \mathbf{A} , a basis for which is computed to be

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Any allocation of vortex strengths that is a linear combination of these basis vectors is a relative equilibrium configuration on the cube. We summarize this result in the following proposition.

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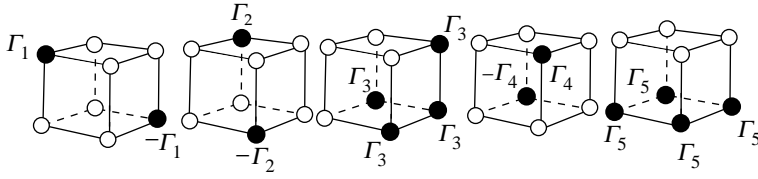


Figure 6. The five basis elements of the cube. Any linear combination of these forms an equilibrium. Open circles represent a vortex of zero strength.

Proposition 5.1. *Label the vortices on the cube and their strengths according to figure 5. The set of relative equilibria on the cube corresponds to an assignment of vortex strengths $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8)^T$ with $\Gamma \in \mathcal{N}(\mathbf{A})$, or equivalently $\Gamma \in \text{Range}(\mathbf{B})$ with \mathbf{A} and \mathbf{B} ,*

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix}.$$

The five basis elements of the cube are shown in figure 6. Any linear combination of these gives rise to an equilibrium configuration.

6. The icosahedron

The icosahedron can be best be viewed as 12 vertices lying on three mutually orthogonal golden ratio rectangles as shown in figure 7. Consider, for example, the equation for the edge l_{12}

$$\pi \frac{dl_{12}^2}{dt} = \sum_k^n \Gamma_k [\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_k)] \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right]. \tag{6.1}$$

There are 10 interaction terms and we show the first six to be automatically zero.

$$\begin{aligned} \Gamma_3 \mathbf{x}_3 \cdot (\mathbf{x}_1 \times \mathbf{x}_2) \left[\frac{1}{l_{13}^2} - \frac{1}{l_{23}^2} \right] &= \rho_1^3 \Gamma_3 \begin{bmatrix} 0 \\ 1 \\ -\phi \end{bmatrix} \cdot \begin{vmatrix} i & j & k \\ 0 & -1 & \phi \\ 0 & 1 & \phi \end{vmatrix} \left[\frac{1}{l_{13}^2} - \frac{1}{l_{23}^2} \right] \\ &= \rho_1^3 \Gamma_3 \begin{bmatrix} 0 \\ 1 \\ -\phi \end{bmatrix} \cdot \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \left[\frac{1}{l_{13}^2} - \frac{1}{l_{23}^2} \right] = 0, \end{aligned}$$

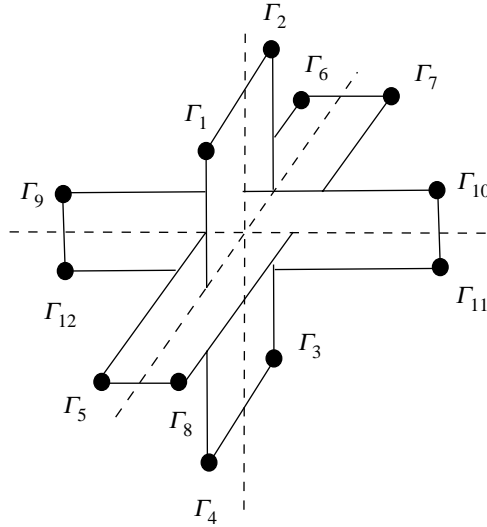


Figure 7. The icosahedron viewed as 12 vertices arranged on three mutually orthogonal golden rectangles. The vortices are located as indicated with the golden ratio $\phi = (\sqrt{5} + 1)/2$. Vortex locations are $\mathbf{x}_1 = \rho_1(0, -1, \phi)$, $\mathbf{x}_2 = \rho_1(0, 1, \phi)$, $\mathbf{x}_3 = \rho_1(0, 1, -\phi)$, $\mathbf{x}_4 = \rho_1(0, -1, -\phi)$, $\mathbf{x}_5 = \rho_1(-1, -\phi, 0)$, $\mathbf{x}_6 = \rho_1(-1, \phi, 0)$, $\mathbf{x}_7 = \rho_1(1, \phi, 0)$, $\mathbf{x}_8 = \rho_1(1, -\phi, 0)$, $\mathbf{x}_9 = \rho_1(-\phi, 0, 1)$, $\mathbf{x}_{10} = \rho_1(\phi, 0, 1)$, $\mathbf{x}_{11} = \rho_1(\phi, 0, -1)$, $\mathbf{x}_{12} = \rho_1(-\phi, 0, -1)$ where $\rho_1 \equiv \sqrt{2/(5 + \sqrt{5})}$ normalizes each vector to have unit length.

$$\begin{aligned} \Gamma_4 \mathbf{x}_4 \cdot (\mathbf{x}_1 \times \mathbf{x}_2) \left[\frac{1}{l_{14}^2} - \frac{1}{l_{24}^2} \right] &= \rho_1^3 \Gamma_4 \begin{bmatrix} 0 \\ -1 \\ -\phi \end{bmatrix} \cdot \begin{vmatrix} i & j & k \\ 0 & -1 & \phi \\ 0 & 1 & \phi \end{vmatrix} \left[\frac{1}{l_{14}^2} - \frac{1}{l_{24}^2} \right] \\ &= \rho_1^3 \Gamma_4 \begin{bmatrix} 0 \\ -1 \\ -\phi \end{bmatrix} \cdot \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \left[\frac{1}{l_{14}^2} - \frac{1}{l_{24}^2} \right] = 0, \end{aligned}$$

$$\Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_2) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right] = 0 \quad \text{as} \quad [l_{1k}^2 = l_{2k}^2] \quad (k = 9, 10, 11, 12).$$

We now look at the remaining four terms in the equation for l_{12}

$$\begin{aligned} \Gamma_k \mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_k) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right] &= 2\rho_1^3 \phi \Gamma_k \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{2k}^2} \right] \\ &= (-1)^k 2\rho_1^3 d_{21k} \phi \Gamma_k, \quad (k = 5, 6, 7, 8). \end{aligned}$$

The simplifications mentioned in §1 are again apparent in the common term d_{21k} , which arises as a result of the inherent symmetry of the icosahedron. The term is common to the previous four equations. With reference to figure 7 and the

coordinates of the vertices, it is clear that the common term d_{21k} is

$$d_{21k} = \frac{1}{l_{2k}^2} - \frac{1}{l_{1k}^2} = \frac{1}{2(1 + \phi + \phi^2)} - \frac{1}{2(1 - \phi + \phi^2)}.$$

Putting these together simplifies the equation for the edge l_{12}

$$\pi \frac{dl_{12}^2}{dt} = \rho_1^3 (-2d_{21k}\phi\Gamma_5 + 2d_{21k}\phi\Gamma_6 - 2d_{21k}\phi\Gamma_7 + 2d_{21k}\phi\Gamma_8).$$

So a necessary condition for a relative equilibrium is

$$\pi \frac{dl_{12}^2}{dt} = 2d_{21k}\phi\rho_1^3 (-\Gamma_5 + \Gamma_6 - \Gamma_7 + \Gamma_8) = 0 \Rightarrow -\Gamma_5 + \Gamma_6 - \Gamma_7 + \Gamma_8 = 0. \tag{6.2}$$

The analysis can be repeated for all 30 edges on the icosahedron simply by permuting the other 29 edges into the edge l_{12} and obtaining the relevant linear equations on the vortex strengths corresponding to the given edge as in equation (6.2). As in the case of the cube, some of these are redundant. Eliminating the redundant equations yields the following reduced system of 15 linear equations for vortex strengths: $\Gamma_1, \dots, \Gamma_{12}$.

$$\mathbf{A}\mathbf{\Gamma} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \\ \Gamma_7 \\ \Gamma_8 \\ \Gamma_9 \\ \Gamma_{10} \\ \Gamma_{11} \\ \Gamma_{12} \end{bmatrix} = 0.$$

The structure of \mathbf{A} yields immediately the solution $\mathbf{\Gamma} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$, the case of all identical vortices. All relative equilibrium solutions are found by finding $\mathcal{N}(\mathbf{A})$ the nullspace (kernel) of \mathbf{A} a basis for which is

computed to be

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Any allocation of vortex strengths that is a linear combination of these basis vectors is a relative equilibrium configuration on the icosahedron. We summarize this result in the following proposition with the matrix \mathbf{A} replaced by its simplified row echelon form.

Proposition 6.1. *Label the vortices on the icosahedron and their strengths according to figure 7. The set of relative equilibria on the icosahedron corresponds to an assignment of vortex strengths $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \dots, \Gamma_{11}, \Gamma_{12})^T$ with $\mathbf{\Gamma} \in \mathcal{N}(\mathbf{A})$, or equivalently $\mathbf{\Gamma} \in \text{Range}(\mathbf{B})$ with \mathbf{A} and \mathbf{B} ,*

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Figure 8 depicts the seven basis elements associated with the icosahedron. Any linear combination of these elements gives rise to an equilibrium configuration.

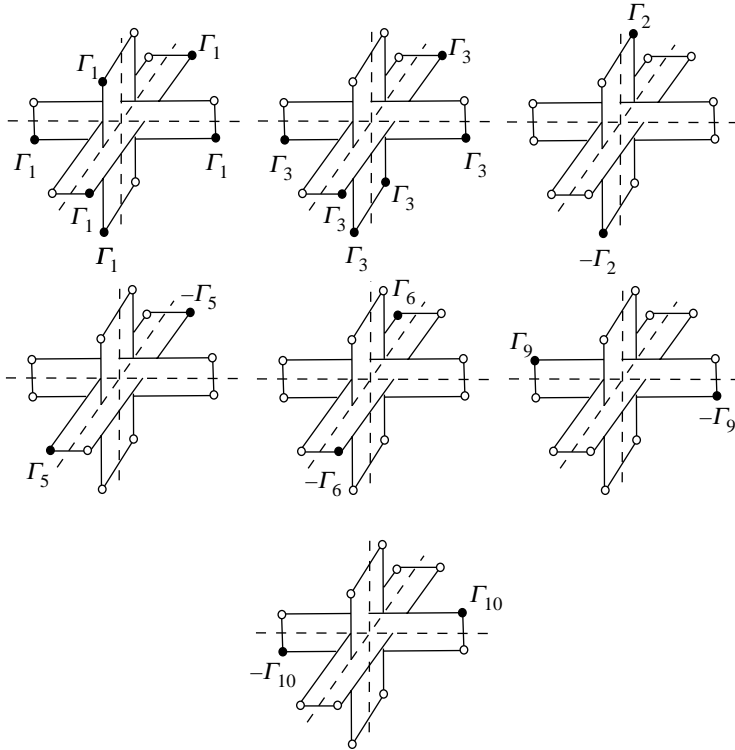


Figure 8. The seven basis elements of the icosahedron corresponding to $v_1, v_2, v_3, v_4, v_5, v_6, v_7$. Any linear combination of these elements give rise to an equilibrium configuration. Open circles represent vortices of zero strength.

7. The dodecahedron

The dodecahedron can best be viewed by regarding it as four planar rings stacked on each other as shown in figure 9. Observe that the faces on the dodecahedron are pentagons, not triangles. Therefore, it does not suffice to look at the relative equations for the edges only. The pentagonal faces have to be triangulated and the relative equations for the diagonals on them have to be considered also. Begin by examining the relative equations for one of the edges such as l_{16} .

$$\pi \frac{dl_{16}^2}{dt} = \sum_k \Gamma_k V_{16k} d_{16k} = \sum_k \Gamma_k [\mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_6)] \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{6k}^2} \right], \tag{7.1}$$

where for the dodecahedron

$$\mathbf{x}_1 \times \mathbf{x}_6 = \rho_2^2 \begin{vmatrix} i & j & k \\ 2 & 0 & \phi + 1 \\ 2\phi & 0 & \phi - 1 \end{vmatrix} = \rho_2^2 \begin{bmatrix} 0 \\ 2(1 + \phi^2) \\ 0 \end{bmatrix}.$$

As there are 20 vortices on the dodecahedron, there are 18 terms to consider in equation (7.1). We examine each of these terms exploiting the symmetry inherent in the dodecahedron. Since $l_{17} = l_{67}, l_{1,10} = l_{6,10}, l_{1,17} = l_{6,17}, l_{1,20} = l_{6,20}$,

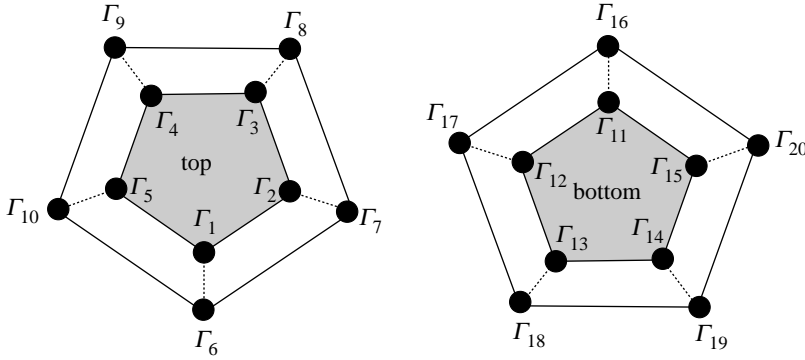


Figure 9. The dodecahedron shown as four planar rings stacked upon each other. Vortex locations are given by $\mathbf{x}_1, \dots, \mathbf{x}_5 = \rho_2(2 \cos(2k\pi/5), 2 \sin(2k\pi/5), \phi + 1), k = 0, \dots, 4$; $\mathbf{x}_6, \dots, \mathbf{x}_{10} = \rho_2(2 \cos(2k\pi/5), 2 \sin(2k\pi/5), \phi - 1), k = 0, \dots, 4$; $\mathbf{x}_{16}, \dots, \mathbf{x}_{20} = -\rho_2(2 \cos(2k\pi/5), 2 \sin(2k\pi/5), \phi - 1), k = 0, \dots, 4$; $\mathbf{x}_{11}, \dots, \mathbf{x}_{15} = -\rho_2(2 \cos(2k\pi/5), 2 \sin(2k\pi/5), \phi + 1), k = 0, \dots, 4$, where ϕ is the golden ratio $\phi = (\sqrt{5} + 1)/2$ and $\rho_2 \equiv 1/\sqrt{\phi^2 + 2\phi + 5}$ is chosen to normalize each vector to have unit length.

we have

$$\Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_6) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{6k}^2} \right] = 0 \quad (k = 7, 10, 17, 20). \tag{7.2}$$

The next two terms are zero because the relevant vortices lie on great circles seen as follows:

$$\Gamma_{11} \mathbf{x}_{11} \cdot (\mathbf{x}_1 \times \mathbf{x}_6) \left[\frac{1}{l_{1,11}^2} - \frac{1}{l_{6,11}^2} \right] = \Gamma_{11} \rho_2^3 \begin{bmatrix} -2 \\ 0 \\ -(\phi + 1) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2(1 + \phi)^2 \\ 0 \end{bmatrix} \left[\frac{1}{l_{1,11}^2} - \frac{1}{l_{6,11}^2} \right] = 0, \tag{7.3}$$

$$\Gamma_{16} \mathbf{x}_{16} \cdot (\mathbf{x}_1 \times \mathbf{x}_6) \left[\frac{1}{l_{1,16}^2} - \frac{1}{l_{6,16}^2} \right] = \Gamma_{16} \rho_2^3 \begin{bmatrix} -2\phi \\ 0 \\ 1 - \phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2(1 + \phi)^2 \\ 0 \end{bmatrix} \left[\frac{1}{l_{1,16}^2} - \frac{1}{l_{6,16}^2} \right] = 0. \tag{7.4}$$

The next four terms have the common term α

$$\Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_6) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{6k}^2} \right] = \alpha (-1)^{k+1} \rho_2^3 \Gamma_k \quad (k = 2, 5, 18, 19). \tag{7.5}$$

It can be shown by symmetry that the next four terms have the common term γ

$$\Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_6) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{6k}^2} \right] = \gamma (-1)^{k+1} \rho_2^3 \Gamma_k \quad (k = 8, 9, 12, 15). \tag{7.6}$$

The next four terms have the common term β

$$\Gamma_k \mathbf{x}_k \cdot (\mathbf{x}_1 \times \mathbf{x}_6) \left[\frac{1}{l_{1k}^2} - \frac{1}{l_{6k}^2} \right] = (-1)^k \beta \rho_2^3 \Gamma_k \quad (k = 3, 4, 13, 14). \tag{7.7}$$

Putting these together simplifies the equation for the edge l_{16}

$$\begin{aligned} \pi \frac{dl_{16}^2}{dt} &= \alpha \rho_2^3 [-\Gamma_2 + \Gamma_5 - \Gamma_{18} + \Gamma_{19}] + \beta \rho_2^3 [-\Gamma_3 + \Gamma_4 - \Gamma_{13} + \Gamma_{14}] \\ &\quad + \gamma \rho_2^3 [-\Gamma_8 + \Gamma_9 - \Gamma_{12} + \Gamma_{15}], \end{aligned}$$

where the constants α, β, γ are calculated as

$$\begin{aligned} \alpha &= \frac{(\phi^2 + 1)\sqrt{4\phi^2 - 1}}{2\phi} \left[\frac{\phi}{2\phi - 1} - \frac{1}{\phi^2 + 1} \right]; \\ \gamma &= \frac{(\phi^2 + 1)\sqrt{4\phi^2 - 1}}{2\phi} \left[\frac{1}{(2\phi^2 + 1)} - \frac{1}{4(\phi^2 - 1)} \right]; \quad \alpha \cdot \gamma = 2\beta^2, \end{aligned}$$

with ϕ being the golden ratio $\phi = (\sqrt{5} + 1)/2$. So a necessary condition for a relative equilibrium is

$$\begin{aligned} \pi \frac{dl_{16}^2}{dt} &= \alpha \rho_2^3 [-\Gamma_2 + \Gamma_5 - \Gamma_{18} + \Gamma_{19}] + \beta \rho_2^3 [-\Gamma_3 + \Gamma_4 - \Gamma_{13} + \Gamma_{14}] \\ &\quad + \gamma \rho_2^3 [-\Gamma_8 + \Gamma_9 - \Gamma_{12} + \Gamma_{15}] = 0, \end{aligned}$$

or the linear equation on the vortex strengths

$$\alpha [-\Gamma_2 + \Gamma_5 - \Gamma_{18} + \Gamma_{19}] + \beta [-\Gamma_3 + \Gamma_4 - \Gamma_{13} + \Gamma_{14}] + \gamma [-\Gamma_8 + \Gamma_9 - \Gamma_{12} + \Gamma_{15}] = 0. \tag{7.8}$$

The analysis can be repeated for all 30 edges on the dodecahedron simply by permuting the remaining edges onto the edge l_{16} and obtaining the relevant linear equations on the vortex strengths corresponding to the given edge as in equation (7.8). As in the case of the icosahedron some of these are redundant. For example, our choice of vortex labelling gives an equation for $d_{1,6}^2/dt$ that is the same as $d_{11,16}^2/dt$ with the following equivalency $\{1 \leftrightarrow 11, 2 \leftrightarrow 12, \dots, 9 \leftrightarrow 19, 10 \leftrightarrow 20\}$. Eliminating the redundant equations yields the following system of 15 linear equations

$$A\Gamma = \mathbf{0} \tag{7.9}$$

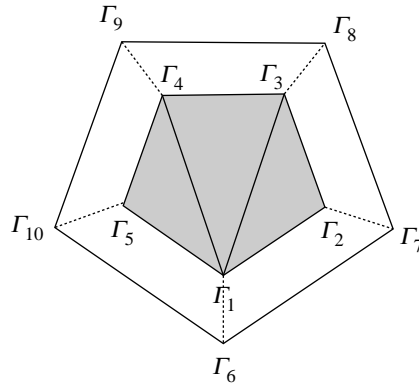


Figure 10. A triangulation of one of the pentagonal faces of the dodecahedron, showing dodecahedral edges as well as chords that form the sides of the triangles. We consider the chords l_{13} and l_{14} .

on the vortex strengths $\mathbf{\Gamma}^T = [\Gamma_1, \dots, \Gamma_{20}]$, where the matrix \mathbf{A} , is

$$\mathbf{A} = \begin{bmatrix} 0 & -\alpha - \beta & \beta & \alpha & 0 & 0 & -\gamma & \gamma & 0 & 0 & -\gamma - \beta & \beta & \gamma & 0 & 0 & -\alpha & \alpha & 0 \\ -\alpha - \beta & \beta & \alpha & 0 & 0 & -\gamma & \gamma & 0 & 0 & -\gamma - \beta & \beta & \gamma & 0 & 0 & -\alpha & \alpha & 0 & 0 \\ -\beta & \beta & \alpha & 0 & -\alpha - \gamma & \gamma & 0 & 0 & 0 & -\beta & \beta & \gamma & 0 & -\gamma - \alpha & \alpha & 0 & 0 & 0 \\ \beta & \alpha & 0 & -\alpha - \beta & \gamma & 0 & 0 & 0 & -\gamma & \beta & \gamma & 0 & -\gamma - \beta & \alpha & 0 & 0 & 0 & -\alpha \\ \alpha & 0 & -\alpha - \beta & \beta & 0 & 0 & 0 & -\gamma & \gamma & \gamma & 0 & -\gamma - \beta & \beta & 0 & 0 & 0 & -\alpha & \alpha \\ 0 & 0 & \alpha & 0 & -\alpha & \alpha & -\alpha & \beta & 0 & -\beta & 0 & 0 & \gamma & 0 & -\gamma & \gamma & -\gamma & \beta & 0 & -\beta \\ 0 & \alpha & 0 & -\alpha & 0 & -\alpha & \beta & 0 & -\beta & \alpha & 0 & \gamma & 0 & -\gamma & 0 & -\gamma & \beta & 0 & -\beta & \gamma \\ \alpha & 0 & -\alpha & 0 & 0 & \beta & 0 & -\beta & \alpha & -\alpha & \gamma & 0 & -\gamma & 0 & 0 & \beta & 0 & -\beta & \gamma & -\gamma \\ 0 & -\alpha & 0 & 0 & \alpha & 0 & -\beta & \alpha & -\alpha & \beta & 0 & -\gamma & 0 & 0 & \gamma & 0 & -\beta & \gamma & -\gamma & \beta \\ -\alpha & 0 & 0 & \alpha & 0 & -\beta & \alpha & -\alpha & \beta & 0 & -\gamma & 0 & 0 & \gamma & 0 & -\beta & \gamma & -\gamma & \beta & 0 \\ -\alpha & -\beta & -\gamma & 0 & 0 & 0 & \beta & 0 & \gamma & \alpha & -\gamma - \beta & -\alpha & 0 & 0 & 0 & \beta & 0 & \alpha & \gamma \\ 0 & -\alpha & -\beta & -\gamma & 0 & \alpha & 0 & \beta & 0 & \gamma & 0 & -\gamma - \beta & -\alpha & 0 & \gamma & 0 & \beta & 0 & \alpha \\ 0 & 0 & -\alpha & -\beta & -\gamma & \gamma & \alpha & 0 & \beta & 0 & 0 & 0 & -\gamma - \beta & -\alpha & \alpha & \gamma & 0 & \beta & 0 \\ -\gamma & 0 & 0 & -\alpha - \beta & 0 & \gamma & \alpha & 0 & \beta & -\alpha & 0 & 0 & -\gamma - \beta & 0 & \alpha & \gamma & 0 & \beta \\ -\beta & -\gamma & 0 & 0 & -\alpha & \beta & 0 & \gamma & \alpha & 0 & -\beta & -\alpha & 0 & 0 & -\gamma & \beta & 0 & \alpha & \gamma & 0 \end{bmatrix}.$$

We remark that this system yields only necessary conditions on the vortex strengths for a relative equilibrium on the dodecahedron as they arise from looking only at the dynamical equations for the edge lengths. The faces on the dodecahedron are *not* triangles, so it remains to triangulate the pentagonal faces and also consider the relative equations for chords belonging to these triangulations. A typical triangulation of one of the pentagonal faces of the dodecahedron is shown in figure 10. Observe that there are 12 pentagonal faces

on the dodecahedron so that, in principle, the dynamical equations for 24 chord lengths have to be examined. We will see though that (as was the case with the 30 edges with only 15 non-redundant equations), symmetry yields that the equations for only 12 chords are non-redundant.

As with the edges of the dodecahedron, by exploiting symmetry, we can find the equations describing the evolution of the chords l_{13} and l_{14} . Omitting the details, we obtain

$$\begin{aligned} \pi \frac{dl_{13}^2}{dt} = & a\rho_2^3 \left[\Gamma_5 - \Gamma_4 + \frac{\Gamma_8}{2} - \frac{\Gamma_6}{2} \right] + b\rho_2^3 [\Gamma_{10} - \Gamma_9 + \Gamma_{20} - \Gamma_{19}] \\ & + c\rho_2^3 \left[\Gamma_{18} - \Gamma_{16} + \frac{\Gamma_{15}}{2} - \frac{\Gamma_{14}}{2} \right], \end{aligned} \tag{7.10}$$

$$\begin{aligned} \pi \frac{dl_{14}^2}{dt} = & a\rho_2^3 \left[\Gamma_3 - \Gamma_2 + \frac{\Gamma_6}{2} - \frac{\Gamma_9}{2} \right] + b\rho_2^3 [\Gamma_8 - \Gamma_7 + \Gamma_{18} - \Gamma_{17}] \\ & + c\rho_2^3 \left[\Gamma_{16} - \Gamma_{19} + \frac{\Gamma_{13}}{2} - \frac{\Gamma_{12}}{2} \right], \end{aligned} \tag{7.11}$$

where it can be calculated that

$$a = \alpha\phi, \quad c = \frac{a}{\left(2\phi - \frac{1}{\phi}\right)^2}, \quad b = \frac{a + c}{3}.$$

To see explicitly why the equations for only 12 chords are necessary consider the equations for the ‘mirror’ chords $l_{11,13}$ and $l_{11,14}$

$$\begin{aligned} \pi \frac{dl_{11,13}^2}{dt} = & a\rho_2^3 \left[\Gamma_{15} - \Gamma_{14} + \frac{\Gamma_{18}}{2} - \frac{\Gamma_{16}}{2} \right] + b\rho_2^3 [\Gamma_{20} - \Gamma_{19} + \Gamma_{10} - \Gamma_9] \\ & + c\rho_2^3 \left[\Gamma_8 - \Gamma_6 + \frac{\Gamma_5}{2} - \frac{\Gamma_4}{2} \right], \end{aligned} \tag{7.12}$$

$$\begin{aligned} \pi \frac{dl_{11,14}^2}{dt} = & a\rho_2^3 \left[\Gamma_{13} - \Gamma_{12} + \frac{\Gamma_{16}}{2} - \frac{\Gamma_{19}}{2} \right] + b\rho_2^3 [\Gamma_{18} - \Gamma_{17} + \Gamma_8 - \Gamma_7] \\ & + c\rho_2^3 \left[\Gamma_{16} - \Gamma_9 + \frac{\Gamma_3}{2} - \frac{\Gamma_2}{2} \right]. \end{aligned} \tag{7.13}$$

Setting to zero, equation (7.10) and its ‘mirror’ equations and eliminating the redundant equations yields the following system of 12 linear equations

$$BF = \mathbf{0}, \tag{7.14}$$

on the vortex strengths $\mathbf{\Gamma}^T = [\Gamma_1, \dots, \Gamma_{20}]$, where the matrix \mathbf{B} , is

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & -a & a & -\frac{a}{2} & 0 & \frac{a}{2} & -b & b & 0 & 0 & 0 & -\frac{c}{2} & \frac{c}{2} & -c & 0 & c & -b & b \\ 0 & -a & a & 0 & 0 & \frac{a}{2} & -b & b & -\frac{a}{2} & 0 & 0 & -\frac{c}{2} & \frac{c}{2} & 0 & 0 & c & -b & b & -c & 0 \\ 0 & -\frac{a}{2} & -b & 0 & 0 & a & c & -\frac{c}{2} & b & 0 & 0 & -c & -b & 0 & 0 & \frac{c}{2} & \frac{a}{2} & -a & b & 0 \\ 0 & \frac{a}{2} & -c & -b & -a & 0 & b & 0 & 0 & a & 0 & c & -\frac{a}{2} & -b & -\frac{c}{2} & 0 & b & 0 & 0 & \frac{c}{2} \\ 0 & a & b & c & -\frac{a}{2} & 0 & -a & 0 & 0 & -b & 0 & \frac{c}{2} & b & \frac{a}{2} & -c & 0 & -\frac{c}{2} & 0 & 0 & -b \\ 0 & 0 & 0 & b & \frac{a}{2} & -a & 0 & -b & \frac{c}{2} & -c & 0 & 0 & 0 & b & c & -\frac{c}{2} & 0 & -b & a & -\frac{a}{2} \\ -\frac{a}{2} & 0 & a & b & c & -b & 0 & -a & 0 & 0 & -c & 0 & \frac{c}{2} & b & \frac{a}{2} & -b & 0 & -\frac{c}{2} & 0 & 0 \\ \frac{a}{2} & 0 & 0 & 0 & b & -c & -a & 0 & -b & \frac{c}{2} & c & 0 & 0 & 0 & b & -\frac{a}{2} & -\frac{c}{2} & 0 & -b & a \\ c & -\frac{a}{2} & 0 & a & b & 0 & -b & 0 & -a & 0 & \frac{a}{2} & -c & 0 & \frac{c}{2} & b & 0 & -b & 0 & -\frac{c}{2} & 0 \\ b & \frac{a}{2} & 0 & 0 & 0 & \frac{c}{2} & -c & -a & 0 & -b & b & c & 0 & 0 & 0 & a & -\frac{a}{2} & -\frac{c}{2} & 0 & -b \\ b & c & -\frac{a}{2} & 0 & a & 0 & 0 & -b & 0 & -a & b & \frac{a}{2} & -c & 0 & \frac{c}{2} & 0 & 0 & -b & 0 & -\frac{c}{2} \\ 0 & b & \frac{a}{2} & 0 & 0 & -b & \frac{c}{2} & -c & -a & 0 & 0 & b & c & 0 & 0 & -b & a & -\frac{a}{2} & -\frac{c}{2} & 0 \end{bmatrix}.$$

Necessary and sufficient conditions for relative equilibria are obtained by finding solutions $\mathbf{\Gamma}$ to both the equations for the edges, equation (7.9), and the equations for the chords coming from the triangulations, equation (7.14). We write this as the augmented matrix system

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \mathbf{\Gamma} = \mathbf{0}. \tag{7.15}$$

As with the icosahedron, the structure of \mathbf{A} and \mathbf{B} yields immediately the solution $\mathbf{\Gamma} = (1, 1, \dots, 1, 1)^T$, the case of all identical vortices. All relative equilibrium solutions are found by finding the nullspace (kernel) of the augmented matrix in (7.15) a basis for which is computed to be

$$\mathbf{v}_1 = [-\phi, -\phi, -\phi, -\phi, -\phi, -\phi + 1, -\phi + 1, -\phi + 1, -\phi + 1, -\phi + 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0]^T,$$

$$\mathbf{v}_2 = [1, \phi - 1, 1, \phi, \phi, \phi - 1, 0, \phi - 1, \phi, \phi, \phi - 1, 1, \phi - 1, 0, 0, 1, \phi, 1, 0, 0]^T,$$

$$\mathbf{v}_3 = [0, 0, -1, -\phi, -1, \phi - 1, \phi - 1, -1, -2, -1, -1, -1, 0, \phi - 1, 0, -\phi, -\phi, 0, 1, 0]^T,$$

$$\mathbf{v}_4 = [\phi, 2, \phi + 1, \phi + 1, 2, 2 - \phi, 1, 2, 2, 1, 2 - \phi, 0, -\phi + 1, -\phi + 1, 0, \phi, 1, 0, 0, 1]^T.$$

Any allocation of vortex strengths that is a linear combination of these basis vectors is a relative equilibrium configuration on the dodecahedron. We summarize this result in the following proposition.

Proposition 7.1. *Label the vortices on the dodecahedron and their strengths according to figure 9. The set of relative equilibria on the dodecahedron corresponds to an assignment of vortex strengths $\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_{19}, \Gamma_{20})^T$ for $\Gamma \in \text{Range}(C)$ with C ,*

$$C = \begin{bmatrix} -\phi & 1 & 0 & \phi \\ -\phi & \phi - 1 & 0 & 2 \\ -\phi & 1 & -1 & \phi + 1 \\ -\phi & \phi & -\phi & \phi + 1 \\ -\phi & \phi & -1 & 2 \\ \phi + 1 & \phi - 1 & \phi - 1 & 2 - \phi \\ \phi + 1 & 0 & \phi - 1 & 1 \\ \phi + 1 & \phi - 1 & -1 & 2 \\ \phi + 1 & \phi & -2 & 2 \\ \phi + 1 & \phi & -1 & 1 \\ 1 & \phi - 1 & -1 & 2 - \phi \\ 1 & 1 & -1 & 0 \\ 1 & \phi - 1 & 0 & -\phi + 1 \\ 1 & 0 & \phi - 1 & -\phi + 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -\phi & \phi \\ 0 & \phi & -\phi & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The four basis elements associated with the dodecahedron are shown in figure 11.

8. Conclusion

The method described in this paper holds great promise for constructing heterogeneous equilibria for more general geometric configurations on the sphere and should also work well for the planar N -vortex problem, as well as for other particle interaction problems, such as those arising in electrostatics where one has freedom to choose the charges associated with each particle (see Saff & Kuijlaars 1997). There is also the general question of *stability* of such structures, about which far less is known. The only stability analysis of the Platonic solid configurations we know of is that of Kurakin (2004), in which all the vortex

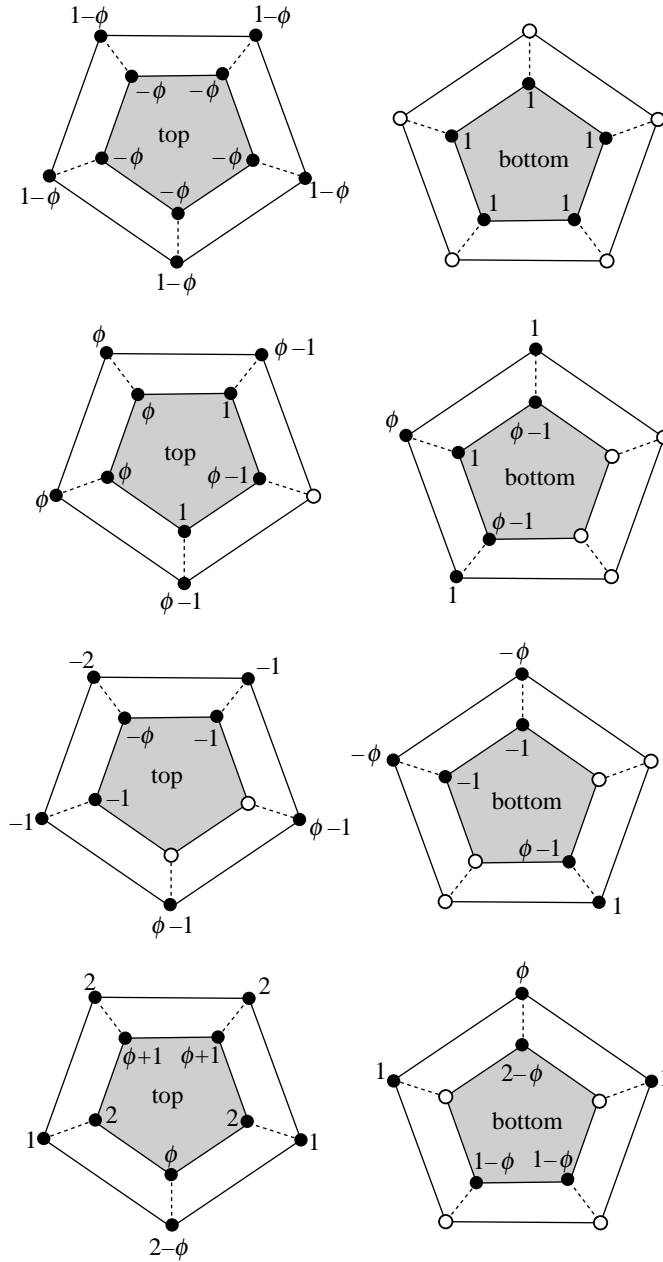


Figure 11. The four basis elements of the dodecahedron. Any linear combination of these forms an equilibrium configuration. Open circles represent vortices of zero strength.

strengths are taken as equal. In this case, the tetrahedron, octahedron and icosahedron are shown to be nonlinearly stable, whereas the cube and dodecahedron are unstable.

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